

# Representable Cartesian Fibrations

Nima Rasekh

4/18/2017

## Motivation

People in geometry are obsessed with objects that classify certain geometric behaviors. Such spaces are commonly called *classifying objects* or *moduli spaces*. A famous example is obviously the space  $BG$ , where  $G$  is a discrete group. It is the classification space for principal  $G$ -bundles.

Let's categorify this story. In the categorified version we wish to have *object-classifiers* devoid of all geometric meaning. More concretely, for a category with finite limits, we want an object  $\Omega$  such that for every object  $c \in \mathcal{C}$  there is an equivalence

$$\mathcal{C}/_c \simeq \text{Hom}_{\mathcal{C}}(c, \Omega)$$

However, this is clearly impossible. First of all the left hand side is a category but the right hand side is a set. But we can think of the left hand side as a set ignoring the morphisms. Still this won't work as in an arbitrary category an object can have non-trivial automorphisms, and so there is no hope of building an equivalence. We could restrict to categories without automorphisms (so called *complete Segal sets*), but that sounds overly restrictive (in particular most common categories have non-trivial automorphisms). Instead, we can restrict to maps that are monomorphisms. The property of a monomorphisms guarantees that we do not have any non-trivial automorphism there. Thus we can hope to find an object  $\Omega$  such that

$$\tau_{-1}(\mathcal{C}/_c) \simeq \text{Hom}_{\mathcal{C}}(c, \Omega)$$

Such an object is called a *subobject classifier* (for obvious reasons).

The study of subobject classifiers is the focus of *elementary topoi*:

**Definition 1.1.** A category  $\mathcal{E}$  is called an elementary topos if it is locally Cartesian closed and it has a subobject classifier.

We can use elementary topoi to study very diverse subjects. On the one side we can restrict to a category of sheaves (if we add a presentability condition)

and thus study all kind of geometric phenomena. On the other sides it gives a model of many type theories including all set theories and thus allows us to construct non-standard models for set theory. For more on elementary topoi see [MM12].

But this is all happening in the world of set theory. We are homotopy theorists and thus have to beef it up to the realm of homotopy theory. Doing so would similiarly allow us to talk about sheaves of spaces (which has already been done in [Lu09] and [Re05] ) but also non-standard models of spaces.

## First steps in Higher Category Theory

We can essentially repeat the exact same question in the realm of higher categories For a higher category with finite limits can we find an object  $\Omega$  such that the following is an equivalence

$$\mathcal{C}_{/c} \simeq \text{map}_{\mathcal{C}}(c, \Omega)$$

where  $\mathcal{C}$  now is a higher category. Now that we are in the realm of higher category we do not have a mapping set, but rather a *mapping space*. Thus we don't have to worry about non-trivial automorphisms. However there is still another problem here. Namely the left hand side is an honest category, whereas the right hand side is just a space. There is a simple way to remedy this, by restricting to the maximal space (called the *core*) inside a higher category. Thus we reduce it to the following question

$$(\mathcal{C}_{/c})^{\text{core}} \simeq \text{map}_{\mathcal{C}}(c, \Omega)$$

This is actually possible and such an object  $\Omega$  is called an *object classifier*.

**Notation 2.1.** Note that now that we have generalized things to the level of object classifiers our construction has set theoretical issues. For the purposes of this talk we will ignore such concerns.

However, using object classifiers doesn't seem like an elegant solution. The more preferable approach would be to increase the amount of data on the right hand side so that we can directly compare it to the over-category itself rather than just the core.

## Complete Segal Spaces

Before we can go there we have to actually fix what we mean by the word *higher category* or else we won't be able to decide how to generalize our construction. But before we can do that some notation:

**Notation 3.1.** Let  $\mathcal{S} = Fun(\Delta^{op}, Set)$  be the category of *spaces* with the Kan model structure.

**Notation 3.2.** Let  $s\mathcal{S} = Fun(\Delta^{op}, \mathcal{S})$  be the category of *simplicial spaces* with the Reedy model structure.

**Definition 3.3.** A *complete Segal space* is a simplicial space  $W$  such that:

1. Is Reedy fibrant: Gives us homotopy invariance
2. Satisfies the Segal condition: Gives us the right category theory
3. Satisfies the completeness condition: Gives us the right homotopy theory

For more details on complete Segal spaces, see the original source [Re01]. With this in mind the left hand side in the equivalence

$$C_{/c} \simeq map_{\mathcal{C}}(c, \Omega)$$

the left hand side is just a simplicial space and the right hand side is a space. Of course they cannot be equivalent!

But now it is quite clear how we can correct it. Instead of taking an object and map into it we should take a *simplicial object*. For any simplicial object  $x_{\bullet} : \Delta^{op} \rightarrow \mathcal{C}$  and object  $c \in \mathcal{C}$  we can get a simplicial space

$$map_{\mathcal{C}}(c, x) =$$

$$map_{\mathcal{C}}(c, x_0) \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} map_{\mathcal{C}}(c, x_1) \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} map_{\mathcal{C}}(c, x_2) \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \dots$$

Now with these new tools we can ask for a simplicial object  $\Omega_{\bullet} : \Delta^{op} \rightarrow \mathcal{C}$  such that there is an equivalence

$$C_{/c} \simeq map(c, \Omega)$$

The problem with all this is that none of this makes much sense or is in any sense precise. We want to think of both sides somehow as functors from  $\mathcal{C}$  which take values in higher categories

$$C_{/-} \simeq map(-, \Omega)$$

In order to make this precise we have to take a long detour into a lot of higher category theory.

So, we have to start where every story about higher categories begins: *Right Fibrations*.

*Remark 3.4.* From here on we always assume that our CSS have finite limits.

## Right Fibrations

Higher category theory gives us a lot of powerful abstractions, however, they come at a heavy price. A lot of constructions don't go through anymore and have to be reworked. In particular, there is no simple way to talk about presheaves, even representable presheaves:

$$\rho_x : \mathcal{C}^{op} \rightarrow \mathcal{S}$$

taking an object  $y$  to  $\text{map}_{\mathcal{C}}(y, x)$ . (If you are bored by this talk then you are more than welcome to try to construct such maps)

In order to fix that we study certain fibrations over  $\mathcal{C}$  which classify the same data. This leads us to following definition:

**Definition 4.1.** A map  $p : R \rightarrow X$  is a *right fibration* if it satisfies following two conditions

1. It is a Reedy fibration
2. The following is a homotopy pullback square

$$\begin{array}{ccc} R_n & \xrightarrow{n^*} & R_0 \\ \downarrow p_n & \lrcorner & \downarrow p_0 \\ X_n & \xrightarrow{n^*} & X_0 \end{array}$$

where the map  $n : [0] \rightarrow [n]$  takes the point to the final object.

*Remark 4.2.* The idea of this definition is that I can lift  $n$ -maps "uniquely up to homotopy" as long as I know where the final vertex maps.

This definition is originally due to Charles Rezk, but was never published. It can also be found in [\[dB16\]](#).

*Remark 4.3.* The idea of a right fibration is that the fiber over each point is the "value" of that functor and the other lifting conditions give the required "functoriality".

Let us see some examples.

**Example 4.4.** Let  $X = *$  be the point. Then a right fibration  $R \rightarrow *$  is just a Kan complex.

*Remark 4.5.* Note that this in particular implies that over every point the fiber of a right fibration is just a space. We think of that space as the "value" of the right fibration at that point.

**Example 4.6.** Let  $X = F(1)$  be the "free arrow". Then the data of a right fibration  $R \rightarrow F(1)$  is the data of three Kan complexes and a zig-zag such that one of the maps is a Kan equivalence

$$\begin{array}{ccc}
 & R_{/01} & \\
 \swarrow & & \searrow \simeq \\
 R_{/0} & & R_{/1}
 \end{array}$$

which is essentially the data of a contravariant map from  $0 \rightarrow 1$ .

**Example 4.7.** Let  $X$  be any Segal space and  $x$  be an object in  $X$ . Then we can construct the Segal space of objects over  $x$  as follows as the following pullback

$$\begin{array}{ccc}
 X_{/x} & \longrightarrow & X^{F(1)} \\
 \downarrow p & \lrcorner & \downarrow \\
 X & \xrightarrow{id \times x} & X \times X
 \end{array}$$

The "forgetful map"  $p : X_{/x} \rightarrow X$  is then a right fibration. It is the right fibration that should correspond to the functor represented by  $x$  as the "value" over any object  $y$  is just the mapping space  $map_{/X}(y, x)$ . Thus we have constructed the *representable right fibration*. This one actually satisfies any nice functoriality condition we might want.

All we have done is construct maps into space, but what we really wanted were maps into higher categories. So, have to again generalize things.

## Cartesian Fibrations

The goal is now to generalize the discussion about right fibrations to a kind of fibrations that can classify maps into higher categories rather than spaces. By now we have discussed what a higher category is, namely a CSS. In order to build a map into a CSS

$$F : \mathcal{C}^{op} \rightarrow CSS$$

it suffices to build a map into simplicial spaces such that the target of each point is exactly a CSS

$$F : \mathcal{C}^{op} \rightarrow s\mathcal{S}$$

but we can think of that as just a simplicial object into spaces

$$F_{\bullet} : \mathcal{C}^{op} \rightarrow \mathcal{S}$$

But what is the fibration analogue for a map into spaces? Yes, exactly, it is just a right fibration. The result of this argument is that the correct notion of a fibration is a simplicial object in the category of right fibrations such that it satisfies the right conditions.

As always before any good definition we need some notations and definitions.

**Definition 5.1.** Let  $ss\mathcal{S} = Fun(\Delta^{op}, s\mathcal{S})$  be the category of bisimplicial spaces.  $ss\mathcal{S}$  has a Reedy model structure

Let  $X$  be a simplicial space. We think of it as a bisimplicial space by defining it as

$$X_{kn} = X_n$$

From now on  $X$  will be a fixed simplicial space embedded in the way stated above.

**Definition 5.2.** A map of bisimplicial spaces  $Y \rightarrow X$  is a *Reedy right fibration* if for each  $k$  the map

$$Y_k \rightarrow X$$

is a right fibration.

A Reedy right fibration classifies maps from a higher category into a simplicial spaces. Now all that is left to do is put the right restrictions on it that guarantees it maps into CSS.

**Definition 5.3.** A *Cartesian fibration* is a map  $Y \rightarrow X$  that satisfies the following three condition

1. It is a Reedy right fibration
2. It satisfies the Segal condition
3. It satisfies the completeness condition

*Remark 5.4.* It is hard to miss the similarity between this definition and the one for complete Segal spaces. This not a coincidence and can be understood under the general context of *complete Segal objects*.

Cartesian fibrations give us the right tool to classify maps into higher categories. Let us see one example:

**Example 5.5.** Let  $\mathcal{C}$  be a CSS with limits. Let  $p : \mathcal{C}^{F(1)} \rightarrow \mathcal{C}$  be the target map. We can easily build a Cartesian fibration out of it. What is the fiber over each point? We have

$$\begin{array}{ccc} C_{/c} & \longrightarrow & \mathcal{C}^{F(1)} \\ \downarrow & \lrcorner & \downarrow p \\ F(0) & \xrightarrow{c} & \mathcal{C} \end{array}$$

so, the Cartesian fibration  $p$  exactly gives us the functor which takes each point  $c$  to the over category  $C_{/c}$ .

The last piece of the puzzle now is the right hand side. We have to be able to construct a Cartesian fibration using the data of a simplicial object. This leads us to the notion of a *representable Cartesian fibration*.

## Representable Cartesian Fibrations

Our goal now is to define a Cartesian fibration that represents a given simplicial object.

Let  $x$  be a simplicial object in  $\mathcal{C}$  then we can define following bisimplicial object over  $\mathcal{C}$ .

$$\mathcal{C}/_{x_0} \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longrightarrow \end{array} \mathcal{C}/_{x_1} \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longrightarrow \end{array} \mathcal{C}/_{x_2} \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longrightarrow \end{array} \cdots$$

This bisimplicial space is a Reedy right fibration by default. But it might not be Cartesian fibration. In order to guarantee that it is a Cartesian fibration we need to have the right conditions on the simplicial object  $x$ , implying it is a *complete Segal object*, which we will not discuss here in greater detail.

Having this definition at hand we can make following definition:

**Definition 6.1.** A Cartesian fibration  $F$  is representable if there exists a complete Segal object  $x$  such that

$$F \simeq C/_x$$

over  $X$ .

We can move on to the main section we wanted to study.

## Elementary Higher Topos

Now after having gone through this long journey we have finally assembled all the pieces we needed. We can now give the definition we have long waited for.

**Definition 7.1.** We call a higher category an *elementary higher topos* if it has finite limits and the Cartesian fibration  $\mathcal{C}^{F(1)} \rightarrow \mathcal{C}$  is representable.

Let me give you one example. Note that for this example I am again going to ignore several set-theoretical issues.

**Example 7.2.** Let  $Sp$  be the complete Segal space of spaces. For any category  $\mathcal{C}$  the core  $\mathcal{C}^{core}$  is actually space. So, we can make following definition:

$$\Omega_n = (Sp^{F(n)})^{core}$$

This way we get a simplicial object  $\Omega_n$  and we have

$$(Sp/_x)_n \simeq maps_{Sp}(x, \Omega_n)$$

In particular

$$(Sp/_x)^{core} \simeq maps_{Sp}(x, \Omega_0)$$

**Example 7.3.** It's actually not too hard to generalize this to the case of topoi in the sense of Lurie and Rezk.

Before we wrap things up let us give an example of how this construction can be used:

**Theorem 7.4.** *Elementary higher topoi are closed under over constructions.*

*Proof.* Let  $\Omega$  be the object classifier of  $\mathcal{C}$  then  $\pi_1 : x \times \Omega \rightarrow x$  is an object classifier for  $\mathcal{C}/x$ .  $\square$

**Theorem 7.5.** *Elementary higher topoi are locally Cartesian closed.*

*Proof.* By the previous theorem it suffices to prove Cartesian closed. The idea of the proof is the following: For two object  $x, y \in \mathcal{C}$  we get a map

$$(x, y) : * \longrightarrow \Omega_0 \times \Omega_0$$

(here  $*$  is the final object). Now we have following pullback square inside  $\mathcal{C}$ :

$$\begin{array}{ccc} \text{map}(x, y) & \longrightarrow & \Omega_1 \\ \downarrow & \lrcorner & \downarrow (s, t) \\ * & \xrightarrow{(x, y)} & \Omega_0 \times \Omega_0 \end{array}$$

Using universal properties we can easily show that  $\text{map}(x, y)$  is the internal mapping object in  $\mathcal{C}$ .  $\square$

**Conjecture 7.6.** *We also have following conjectures:*

1. *Elementary higher topoi have finite colimits.*
2. *Elementary higher topoi satisfy descent.*

## References

- [dB16] P. B. de Brito, *Segal objects and the Grothendieck construction*, arXiv preprint arXiv:1605.00706 (2016).
- [Lu09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies 170, Princeton University Press, Princeton, NJ, 2009, xviii+925 pp. A
- [MM12] S. MacLane, I. Moerdijk. *Sheaves in geometry and Logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.
- [Re01] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math.Soc., 353(2001), no. 3, 973-1007.
- [Re05] C. Rezk, *Toposes and Homotopy Toposes*, (2005). Unpublished notes, <http://www.math.uiuc.edu/~rezk/homotopy-topos-sketch.pdf>