Representable Cartesian Fibrations

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Fibrations not Functoriality

In classical category theory we study functors and presheaves with different values. Examples are functors valued in sets

$$Fun(\mathcal{C}^{op}, \mathcal{S}et)$$

and functors valued in categories

$$Fun(\mathcal{C}^{op}, \mathcal{C}at)$$

We want to take a similar approach in higher category theory, however, there things become quite complicated. That is because in higher category theory there is often not a unique notion of composition. Rather, for two maps, there is a contractible space of maps. Thus defining functors are tricky as they need a lot of coherence data. Therefore, we often take an alternative approach in higher category theory, which means we use fibrations.

$$
\begin{array}{ccc}
F_c & \longrightarrow & \mathcal{D} \\
\downarrow & & \downarrow{\scriptstyle p \,| \, fibration} \\
* & \longrightarrow & \mathcal{C}
\end{array}
$$

with $Value$ arrow into F_c and corner mark $\ulcorner$

So, the slogan is:

"Fibrations not Functors"

Example 1.1. One example is the notion of a *right fibration* (which will be discussed later). A classic example of a right fibration is the over category $\mathcal{C}_{/c} \to \mathcal{C}$. Looking at the fiber we have

$$\begin{array}{ccc}
map_{/\mathcal{C}}(d,c) & \longrightarrow & \mathcal{C}_{/c} \\
\downarrow & \ulcorner & \downarrow \\
* & \xrightarrow{\quad p \quad} & \mathcal{C}
\end{array}$$

The fiber is exactly the value we associate to the representable functors. Thus, we think of this right fibration as the *representable right fibration*.

Cartesian Fibrations

Our next goal is to study fibrations which model presheaves valued in higher categories $\mathcal{C}at_\infty$. Before I move on I should point out this has been studied extensively by Lurie (Section 2.4 of [Lu09]) in the context of quasicategories. It has also been studied by Riehl and Verity ([RV17]) in their model independent framework of ∞-cosmoi and by de Brito using Segal objects ([dB16]).

Before we give any definition let us review the construction of *complete Segal spaces* as introduced by Charles Rezk in [Re01]. Complete Segal spaces are a *model* of higher categories. They come with a simplicial model structure, which gives us efficient ways to study higher categories.

Category	Functorial over $\mathcal{C}$
Spaces $= Fun(\Delta^{op}, \mathcal{S}et) = \mathcal{S}$	Simplicial Spaces over $\mathcal{C} = s\mathcal{S}_{/\mathcal{C}}$
Kan complexes (Kan)	Right fibration (contravariant)
Simplicial Spaces $= Fun(\Delta^{op}, \mathcal{S}) = s\mathcal{S}$	Bisimplicial Spaces over $\mathcal{C} = ss\mathcal{S}_{/\mathcal{C}}$
Reedy fibrant (Reedy)	Reedy right fibration (Reedy contravariant)
Simplicial Spaces $= s\mathcal{S}_{CSS}$	Bisimplicial Spaces over $= (ss\mathcal{S}_{/\mathcal{C}})_{Cart}$
Complete Segal Spaces (CSS)	Cartesian Fibrations (Cartesian)

Let us make this intuitive story given here precise and give concrete definitions. The next definition is by Charles Rezk but also appears in [dB16].

Definition 2.1. Let $\mathcal{C}$ be a CSS. A map $\mathcal{D} \to \mathcal{C}$ is a *right fibration* if it is a Reedy fibration and the following is a homotopy pullback

$$\begin{array}{ccc}
\mathcal{D}_1 & \longrightarrow & \mathcal{D}_0 \\
\downarrow & \ulcorner & \downarrow \\
\mathcal{C}_1 & \longrightarrow & \mathcal{C}_0
\end{array}$$

in the Kan model structure.

Remark 2.2. This definition is quite non-intuitive. But it helps to see what happens when the base is a point. In that case the map $\mathcal{D}_1 \to \mathcal{D}_0$ is an equivalence. As $\mathcal{D}$ is a CSS this means $\mathcal{D}$ is a higher groupoid, so just a space. Thus the fiber over any right fibration is just a space, which is exactly what we expected.

Let's generalize this

Definition 2.3. Let $\mathcal{C}$ be a CSS. A map of *bisimplicial spaces* $\mathcal{R} \to \mathcal{C}$ is a Reedy right fibration if it is a bisimplicial Reedy fibration and it is a level-wise right fibration, meaning that for each k the map $\mathcal{R}_k \to \mathcal{C}$ is a right fibration
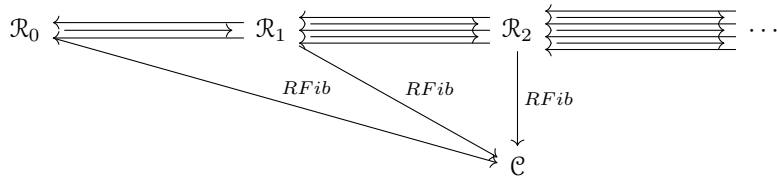
Remark 2.4. As expected a Reedy right fibration models presheaves valued in Reedy fibrant simplicial spaces. This can be checked by looking at Reedy right fibrations over the point.

Definition 2.5. A Reedy right fibration $\mathcal{R} \to \mathcal{C}$ is a *Cartesian fibration* if it satisfies the

1. *Segal Condition:* The maps

$$\mathcal{R}_k \to \mathcal{R}_1 \times_{\mathcal{R}_0} ... \times_{\mathcal{R}_0} \mathcal{R}_1$$

   are Reedy equivalences for $k \geq 2$.

2. *Completeness Condition:* The map

$$\mathcal{R}_3 \times_{\mathcal{R}_1 \times_{\mathcal{R}_0} \mathcal{R}_1 \times_{\mathcal{R}_0} \mathcal{R}_1} \mathcal{R}_1 \to \mathcal{R}_0$$

   is a Reedy equivalence.

Remark 2.6. These two definitions come with a simplicial model structure on the category $ss\mathcal{S}_{/\mathcal{C}}$.

Why?

Instead of now giving various theorems about this definition of a Cartesian fibration I will focus on reasons why it is interesting to study Cartesian fibrations using this new perspective.

(1) Separating Functoriality from the Values of the Functor: The process we described first makes our fibration into a functor valued in simplicial spaces and then modifies the image accordingly. This brings several benefits just by itself.

I It can make the definition of a Cartesian fibration more intuitive. If we understand right fibrations and complete Segal spaces, then this definition is a straightforward combination of those two.

II It can be helpful in concrete computations. For example, when trying to build a Cartesian fibration out of arbitrary maps, we can split the tasks into more manageable pieces, namely one which makes a map into Reedy right fibration and then one which makes Reedy right fibration into a Cartesian fibration.

III It can be generalized to presheaves valued in other categories. In particular, we can model presheaves valued in Segal spaces just by dropping the completeness condition from our Cartesian fibrations. Depending on need we should also be able to model presheaves valued in other categories.

IV The level-wise nature allows us to generalize theorems about right fibrations to the Cartesian setting. For example in right fibrations we following theorem

Theorem 3.1. *A map of simplicial spaces $\mathcal{D} \to \mathcal{E}$ over $\mathcal{C}$ is a contravariant equivalence if and only if for every object $c \in \mathcal{C}$ the map*

$$\mathcal{C}_{c/} \times_{\mathcal{C}} \mathcal{D} \to \mathcal{C}_{c/} \times_{\mathcal{C}} \mathcal{E}$$

is a diagonal Kan equivalence (the diagonal is a Kan equivalences).

Using the definitions above we can generalize that to following theorem

Theorem 3.2. *A map of bisimplicial spaces $\mathcal{D} \to \mathcal{E}$ over $\mathcal{C}$ is a Cartesian equivalence if and only if for every object $c \in \mathcal{C}$ the map*

$$\mathcal{C}_{c/} \times_{\mathcal{C}} \mathcal{D} \to \mathcal{C}_{c/} \times_{\mathcal{C}} \mathcal{E}$$

is a diagonal CSS equivalence (the diagonal is a CSS equivalences).

(2) Representability: Having these tools we can define representable Reedy right fibrations and representable Cartesian Fibrations.

Definition 3.3. Let $\mathcal{C}$ be a CSS. A simplicial object is a map $x_\bullet : \mathcal{N}(\Delta^{op}) \to \mathcal{C}$. The collection of those form a CSS $s\mathcal{C}$.

Theorem 3.4. *For every simplicial object $x_\bullet$ we can define a Reedy right fibration, denoted by $\mathcal{C}_{/x_\bullet}$ that is level-wise equivalent to the representable right fibration $\mathcal{C}_{/x_k}$.*

Remark 3.5. Looking at the fiber over c we get the simplicial space

$$map_{/\mathcal{C}}(y, x_0) \Longleftarrow map_{/\mathcal{C}}(y, x_1) \Lleftarrow map_{/\mathcal{C}}(y, x_2) \Lleftarrow \cdots$$

This definition comes with its own Yoneda lemma

Theorem 3.6. *For two simplicial objects $x_\bullet$ and $y_\bullet$, there is an equivalence*

$$Map_{/\mathcal{C}}(\mathcal{C}_{/x_\bullet}, \mathcal{C}_{/y_\bullet}) \xrightarrow{\simeq} map_{/s\mathcal{C}}(x_\bullet, y_\bullet)$$

Remark 3.7. The Reedy right fibration might not be a Cartesian fibration. For that we need extra conditions on the simplicial object.

Definition 3.8. Let $\mathcal{C}$ be a CSS with finite limits. A *complete Segal object* is a simplicial object that satisfies the complete Segal condition.

Theorem 3.9. *The Reedy right fibration $X_{/x_\bullet}$ is a Cartesian fibration if and only if $x_\bullet$ is a CSO.*

Obviously the Yoneda lemma still holds in this restricted case.

So, what is all of this good for?

I First of all it gives us an effective tool to study complete Segal objects. As an example if I want to define an adjunction of complete Segal objects I can just define it as following diagram

$$
\begin{array}{ccccc}
\mathcal{C}_{/x_\bullet} & \longrightarrow & \mathcal{M} & \longleftarrow & \mathcal{C}_{/x_\bullet} \\
\downarrow & \ulcorner & \downarrow & \urcorner & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{C} \times F(1) & \longleftarrow & \mathcal{C}
\end{array}
$$

where the middle map is a Cartesian and coCartesian fibration and the squares are homotopy pullback squares.

II Moreover, it allows us to talk about "representable functors" in a more advanced way than was possible before. For example, if $\mathcal{C}$ is a CSS with finite limits then the map $t : \mathcal{C}^{F(1)} \to \mathcal{C}$ is a Cartesian fibration (here I am avoiding bisimplicial notation on purpose to simplify things). It represents the functor

$$\mathcal{C}^{op} \to \mathcal{C}at_\infty$$

$$c \longmapsto \mathcal{C}_{/c}$$

We also have a functor that "forgets" all the additional categorical structure and just leaves us with the underlying space, which is called core. Composing those I get a functor

$$\mathcal{C}^{op} \to \mathcal{C}at_\infty \to \mathcal{S}$$

$$c \longmapsto \mathcal{C}_{/c} \to (\mathcal{C}_{/c})^{core}$$

So we get a functor into spaces. Representability of this functor is quite interesting as it gives us a notion of an *object classifier* i.e. an object such that

$$(\mathcal{C}_{/c})^{core} \simeq map(c, \Omega)$$

Notice in order to get here we simplified our functor to be able to represent it in the first place. Having a definition of a representable Cartesian fibration we can talk about representability of the original Cartesian fibration. As object classifiers play a prominent role in topos theory, this might have interesting implications in the study of higher topoi.

5

References

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