Functoriality in Higher Category Theory: Cartesian Fibrations

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The goal of this talk is to study functors for higher categories. We first review functors for classical categories and show how we cannot just generalize our usual approach to functors. At that point we start our discussion of fibrations and show how they allow us to study functors valued in spaces and functors valued in higher categories. All of the material in this talk can be found in more detail in [Ra17a] and [Ra17b].

Categories and Functors

Category theory is based on the idea that it does not suffice to study objects independent of each others, but rather the relations between objects have to be considered as well. Applying that same logic to the study of categories leads to the study of *functors*. In particular, there are two crucial categories $\mathcal{S}et$ and $\mathcal{C}at$ and so we regularly study functors

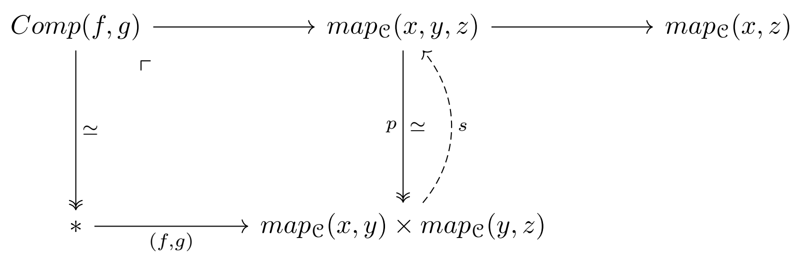
$$\mathcal{C} \to \mathcal{S}et$$

$$\mathcal{C} \to \mathcal{C}at$$

Working with functors can be difficult, but is often still manageable as categories only have mapping sets, and maps of sets are well understood.

Higher Categories and Higher Functors

In the world of higher categories, we study categories "up to homotopy". In particular, this means that for two maps, we do not have a strict composition, but rather a space of compositions. Concretely, for three objects $x, y, z \in \mathcal{C}$ we have following diagram.



So, for any two composable maps f, g, there is a contractible space $Comp(f,g)$ of compositions. This means that managing functoriality is really difficult as we have to keep track of all these spaces of compositions. There are two ways to avoid this problem.

1. Strictify the higher category so that we have strict compositions. In other words, we find a way to make those contractible spaces into points.
2. We find a completely different approach to functoriality that helps us avoid this problem.

In the first approach, we convert the higher category into a category that is enriched over spaces. That way we gain a strict notion of composition. The other one is using the language of fibrations.

Fibrations

In the most general sense, a *fibration* is a map $\mathcal{D} \to \mathcal{C}$ that satisfies some functoriality properties. Here is one of the first examples that arose in the world of classical categories.

Definition 3.1. A functor $p : \mathcal{D} \to \mathcal{C}$ is called *fibration fibered in sets* if for any map $f : a \to b$ in $\mathcal{C}$ and lift $b' \in \mathcal{D}$, there exists a *unique* lift $f' : a' \to b'$ in $\mathcal{D}$.

This fibration plays the role of a functor

$$\mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$$

In this analogy, the value of a point $c \in \mathcal{C}$ is the fiber $p^{-1}(c)$. The condition above guarantees that for each map $f : a \to b$, we get a map of sets $p^{-1}(b) \to p^{-1}(b)$.

There is a way to make this analogy above very precise, via the Grothendieck construction.

Example 3.2. For each functor $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ we get a category $\int_{\mathcal{C}} F$ defined as

- Objects: $(c, t) \in \mathcal{C} \times F(c)$

- Morphisms: A morphisms from (c_1, t_1) to (c_2, t_2) is a map $f : c_1 \to c_2$ such that $F(f)(t_2) = t_1$.

Unsurprisingly we have following fact

Theorem 3.3. *The natural map* $\int_{\mathcal{C}} F \to \mathcal{C}$ *is a category fibered in sets.*

In fact every category fibered in sets can be written in such a format and so the data of such fibration captures the notion of a contravariant functor valued in sets.

The goal is to use the second approach to *define* functoriality rather than prove it.

Right Fibrations

We want to replicate this approach at the level of higher categories. Let $\mathcal{C}$ be a higher category. We want to study maps

$$\mathcal{C}^{op} \to Spaces$$

In order to be able to give concrete definitions we should now specify what we mean by a higher category.

Definition 4.1. A *complete Segal space* is a simplicial space $W : \Delta^{op} \to Spaces \in s\mathcal{S}$ that satisfies following 3 conditions:

1. *Reedy fibrancy*
2. *Segal condition*
3. *Completeness condition*

Remark 4.2. We will not state here what these conditions are, but the one thing that matters is that the conditions are defined as finite limits on simplicial diagram.

We have following important fact about complete Segal spaces.

Theorem 4.3. *A complete Segal space models a higher category. In this way of thinking* W_0 *plays the role of the space of objects,* W_1 *plays the role of space of morphisms,* W_2 *is for compositions.*

Having said that we can finally give our desired definition.

Definition 4.4. A *right fibration* $p : R \to W$ between complete Segal spaces is a Reedy fibration such that the square

$$\begin{array}{ccc} R_1 & \longrightarrow & R_0 \\ \big\downarrow & \ulcorner & \big\downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

is a homotopy pullback square of spaces, meaning that

$$R_1 \to X_1 \underset{X_0}{\times} R_0$$

is a trivial Kan fibration.

This is a direct generalization of the property of fibrations fibered in sets stated above.

Let's see two examples.

Example 4.5. If the base is $X = F(0)$, then a right fibration $R \to F(0)$ is a constant simplicial space. This is consistent with a functor out of a point just being a space.

Example 4.6. If the base is $X = F(1)$, the free arrow, then a right fibration $R \to F(1)$ is the data of a diagram of the form.

$$\begin{array}{ccccc} & & R_{01} & & \\ & \swarrow & & \searrow^{\simeq} & \\ R_0 & & & & R_1 \end{array}$$

Thus we are recovering the notion of a contravariant functor out of the free arrow, as the meaningful data is just the map of spaces $R_{01} \to R_0$.

Just using the definition above we get a lot of expected and important results.

Theorem 4.7. *For a simplicial space X there is a model structure on $s\mathcal{S}_{/X}$, called the contravariant model structure, which satisfies following conditions:*

1. *Fibrant object are right fibrations.*
2. *A map $R \to S$ of right fibrations over X is an equivalence if and only if*

$$R_0 \underset{X_0}{\times} \Delta[0] \to S_0 \underset{X_0}{\times} \Delta[0]$$

   is a Kan equivalence.

3. *A map $f : X \to Y$ gives us a Quillen adjunction*

$$(s\mathcal{S}_{/X})^{contra} \underset{f^*}{\overset{f_!}{\rightleftarrows}} (s\mathcal{S}_{/X})^{contra}$$

which is a Quillen equivalence if f is a categorical equivalence.

Let's see how we can use right fibrations in action.

Example 4.8. One classical fact for classical category theory is that for any object $c \in \mathcal{C}$ there is a functor

$$\mathcal{Y}_c : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$$

that takes an object d to $Hom(d, c)$.

We want to generalize that to the realm of higher categories. For a higher category $\mathcal{C}$ we want a functor

$$\mathcal{Y}_c : \mathcal{C}^{op} \to Spaces$$

that takes a point d to the space $map(d, c)$. Mapping spaces are well-defined, so we can define the values, but functoriality is very difficult to deal with. Here is where right fibrations are a very effective tool.

For each object we have a right fibration $p : \mathcal{C}_{/c} \to \mathcal{C}$. In order to determine which functor it models it suffices to check the fiber over a point

$$\begin{array}{ccc} map(d,c) & \longrightarrow & \mathcal{C}_{/c} \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \longrightarrow & \mathcal{C} \end{array}$$

So, we can think of the over-category projection $\mathcal{C}_{/c} \to \mathcal{C}$ as the "representable right fibration".

Cartesian Fibrations

Until now we have found an effective way to model presheaves valued in spaces. But how about functors

$$\mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_\infty = CSS?$$

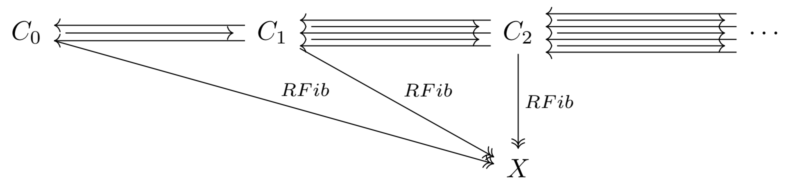
W want to find a notion of fibration that models functors valued in complete Segal spaces. In order to get there we first think of the analogous situation with functors. A functors $F : \mathcal{C}^{op} \to s\mathcal{S}$ is really a simplicial object in functors $F : \mathcal{C}^{op} \to \mathcal{S}$. But we know how to model functors valued in spaces using fibrations, which gives us a clear approach.

Category	Functorial over $\mathcal{C}$
Spaces $= Fun(\Delta^{op}, \mathcal{S}\text{et}) = \mathcal{S}$	Simplicial Spaces over $\mathcal{C} = s\mathcal{S}_{/\mathcal{C}}$
Kan complexes (Kan)	Right fibration (contravariant)
Simplicial Spaces $= Fun(\Delta^{op}, \mathcal{S}) = s\mathcal{S}$	Bisimplicial Spaces over $\mathcal{C} = ss\mathcal{S}_{/\mathcal{C}}$
Reedy fibrant (Reedy)	Reedy right fibration (Reedy contravariant)
Simplicial Spaces $= s\mathcal{S}_{CSS}$	Bisimplicial Spaces over $= (ss\mathcal{S}_{/\mathcal{C}})_{Cart}$
Complete Segal Spaces (CSS)	Cartesian Fibrations (Cartesian)

Using this intuition as a guide we get following definition.

Definition 5.1. Let X be a simplicial and $C \to X$ be a map of bisimplicial spaces. C is a Cartesian fibration over X if it satisfies following three conditions:

1. *Reedy right fibration*:



2. *Segal condition*
3. *Completeness condition*

Using the definition above we can get very similar results to the previous ones.

In particular we can get a new model structure for Cartesian fibrations

Theorem 5.2. *Let X be a simplicial space. There is a model structure on $ss\mathcal{S}_{/X}$, called the Cartesian model structure such that*

1. *Fibrant objects are Cartesian fibrations $C \to X$.*
2. *A map $f : X \to Y$ gives us a Quillen adjunction*

$$(s\mathcal{S}_{/X})^{Cart} \underset{f^*}{\overset{f_!}{\rightleftarrows}} (s\mathcal{S}_{/X})^{Cart}$$

which is a Quillen equivalence if f is a categorical equivalence.

References

[Ra17a] N. Rasekh, *Yoneda Lemma for Simplicial Spaces*, arXiv preprint arXiv:1711.03160 (2017).

[Ra17b] N. Rasekh, *Cartesian Fibrations and Representability*, arXiv preprint arXiv:1711.03670 (2017).