Tensor Thom Talk

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The goal of this talk is to talk about recent paper titled *Thom spectra, higher THH and tensors in ∞-categories* with Bruno Stonek and Gabriel Valenzuela with the goal of reworking some concepts related to THH computations of Thom spectra.

1. What's THH?
2. Why THH?
3. Why a new Approach towards THH?
4. Tensors for ∞-Categories
5. THH of Thom Spectra Revisited
6. Moving beyond Thom Spectra

The goal of this talk is to present an ∞-categorical approach to computing THH of Thom spectra.

What's THH?

Before we go on to THH let us recall the classical version, namely HH.

Definition 1.1. Let A be a commutative k-algebra (k a ring). The *Hochschild complex* is a simplicial abelian group defined as

$$A \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}} A \otimes_k A \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}} A \otimes_k A \otimes_k A \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}} \cdots$$

with

$$d_i(a_0 \otimes ... \otimes a_n) = \begin{cases} a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n & \text{if } i < n \\ a_n a_0 \otimes ... \otimes a_{n-1} & \text{if } i = n \end{cases}$$

and

$$s_i(a_0 \otimes ... \otimes a_n) = a_0 \otimes ... \otimes a_i \otimes 1 \otimes a_{i+1} \otimes ... \otimes a_n$$

The *Hochschild homology* $HH(A)$ is the homology of this complex.

We want to generalize this construction to the world of homotopy theory. A first generalization is due to Bökstedt, using a very technical approach, due to the fact that he didn't have a good symmetric monoidal structure for spectra [Bo85]. A construction analogous to the one above, however, can be found in [EKMM95], where the authors give a good symmetric monoidal structre for spectra and then give following definition for THH.

Definition 1.2. [EKMM95] Let A be an E_∞-R-algebra. Then we can define a simplicial E_∞-R-algebra

$$A \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; A \wedge_R A \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; A \wedge_R A \wedge_R A \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

We define the *topological Hochschild homology* E_∞-R-algebra $THH^R(A)$ as the geometric realization of this simplicial object.

Why THH?

Why should homotopy theorists care about THH? The most important answer is algebraic K-theory. Algberaic K-theory contains a lot of fascinating information and relates to all kinds of topics in homotopy theory. However, it is notoriously difficult to compute. For example we have following classical result by Barrett-Priddy-Quillen.

Theorem 2.1. *The algebraic K-theory of finite pointed sets is the sphere spectrum.*

However, we can use THH to give a reasonable approximation of K. Concretely we have a diagram of the form.

$$\begin{tikzcd} K \arrow[r, "\text{Cyclotomic Trace}"] \arrow[dr, "\text{Dennis Trace}"'] & TC \arrow[d] \\ & THH \end{tikzcd}$$

An important result by Dundas, Goodwillie and McCarthy [DGM12] studies the *cyclotomic trace* (in the connective case) and thus gives us a good understanding of the difference between K and TC, but TC is defined by using cyclotomic structure of THH. Thus the computation of THH is a good first step towards a successful computation of K.

Why a new approach towards THH?

In order to understand why it is valuable to have a modern treatment of THH, we will review work by Schlichtkrull [Sc11]. In his paper "Higher Topological

Hochschild Homology of Thom Spectra" he proves the following theorem (using more modern language).

Theorem 3.1. *Let $f : G \to BGl_1(\mathbb{S})$ be a map of E_∞-groups with $M(f)$ the associated Thom spectrum. Then we have an equivalence of E_∞-rings*

$$THH(M(f)) \simeq Mf \wedge \mathbb{S}[BG].$$

While trying to prove this result Schlichtkrull first points to some very intuitive way to prove it using a simple splitting of an E_∞-space. However, he then points out that he cannot translate his intuition into an actual proof, because the construction would not interact well with the model structure he is using. In particular, the constructions do not preserve cofibrancy. Thus he embarks on a very technical journey to get the desired result.

This is an excellent example where the theory of model categories, which in other places can be quite helpful, is actually becoming a burden. This is why it's useful to take a new approach to these results with the outcome that we can make our initial intuition into an actual proof.

So, how can we generalize our study of THH? The first step is to realize an alternative definition of THH for E_∞-ring spectra.

Theorem 3.2. *[MSV97] Let M be an E_∞-ring. Then we have an equivalence*

$$THH(M) \simeq S^1 \otimes M.$$

Here, $\otimes$ is the tensor product of spaces over E_∞-rings. Thus a good way forward is to carefully study the functor $X \otimes -$ for arbitrary spaces X. This is consistent with work of Schlichtkrull, as he actually uses his technical approach to prove something about $X \otimes M(f)$, which then reduces to the case of THH with $X = S^1$.

Tensors for ∞-Categories

We will now take another look at THH from the perspective of ∞-categories and tensors. Thus, we need to review some concepts about ∞-categories

An ∞-category, also called *quasi-category*, is a simplicial set that satisfies an inner horn-lifting condition that gives it a homotopical notion of composition [Lu09]. All classical categorical concepts, such as adjunctions and limits, have been generalized to the setting of ∞-categories. However, there are certain stronger results that only hold for ∞-categories.

We want to understand how ∞-categories are tensored over each other and we will make an analogy to algebra.

Concept	Algebra	Categories	Higher Categories
Object	Set	Category	∞-Category
Structured Object	Abelian Group	Cocomplete Category + accessible = Locally Presentable Categories	Cocomplete ∞-Category + accessible = Presentable ∞-Categories
Morphism	Group Homomorphism	Colimit Preserving Functor	Colimit Preserving Functor
Symmetric Monoidal	$Hom(G \otimes H, K) =$ $BiLin(G \times H, K)$	$Fun^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) =$ $Fun^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$	$Fun^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) =$ $Fun^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$
Unit = Free Object	$\mathbb{Z} \otimes A \cong A$	$\mathcal{S}\mathrm{et} \otimes \mathcal{C} \simeq \mathcal{C}$	$\mathcal{S} \otimes \mathcal{C} \simeq \mathcal{C}$
Monoid Object	Commutative Ring	Symmetric Monoidal Category	Symmetric Monoidal Category
Unit Multiplicative	$\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ $- \times - : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$	$\mathcal{S}\mathrm{et} \otimes \mathcal{S}\mathrm{et} \simeq \mathcal{S}\mathrm{et}$ $- \times - = \coprod_{-} - : \mathcal{S}\mathrm{et} \times \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$	$\mathcal{S} \otimes \mathcal{S} \simeq \mathcal{S}$ $- \times - = \operatorname{colim}_{(-)}(-) : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$
Modules	$R \otimes M \to M$	$\mathcal{R} \otimes \mathcal{C} \to \mathcal{C}$	$\mathcal{R} \otimes \mathcal{C} \to \mathcal{C}$
Module over Unit	$- \times - : \mathbb{Z} \times A \to A$ $(n, a) \mapsto na$	$\coprod_{-} - : \mathcal{S}\mathrm{et} \times \mathcal{C} \to \mathcal{C}$ $(S, T) \mapsto \coprod_{S} c =$ $\operatorname{colim}(S \to * \xrightarrow{\{c\}} \mathcal{C})$	$\operatorname{colim}_{(-)}(-) : \mathcal{S} \times \mathcal{C} \to \mathcal{C}$ $(X, Y) \mapsto \operatorname{colim}_{X}\{c\} =$ $\operatorname{colim}(X \to * \xrightarrow{\{c\}} \mathcal{C})$
Canonical Module	A k-VS $char(k) = 0 \leftrightarrow$ $\begin{tikzcd} A \times \mathbb{Z} \arrow[r] \arrow[d] & A \\ A \times \mathbb{Q} \arrow[ur, dashed] & \end{tikzcd}$	$\mathcal{C}$ is pointed $\leftrightarrow$ $\begin{tikzcd} \mathcal{C} \times \mathcal{S}\mathrm{et} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{C} \times \mathcal{S}\mathrm{et}_* \arrow[ur, dashed] & \end{tikzcd}$	$\mathcal{C}$ is stable $\leftrightarrow$ $\begin{tikzcd} \mathcal{C} \times \mathcal{S} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{C} \times \mathrm{Sp} \arrow[ur, dashed] & \end{tikzcd}$

Here the right hand column is due to work by [GGN16].

Remark 4.1. We want to focus on the category $\mathrm{Grp}_{E_\infty}(\mathcal{S})$. Then we have following diagram

$$\begin{tikzcd}
\mathcal{S} \times \mathrm{Grp}_{E_\infty}(\mathcal{S}) \arrow[d, "(-)_+ \times id"'] \arrow[dr, "- \otimes -"] & \\
\mathcal{S}_* \times \mathrm{Grp}_{E_\infty}(\mathcal{S}) \arrow[r, "- \odot -"] \arrow[d, "F \times id"'] & \mathrm{Grp}_{E_\infty}(\mathcal{S}) \\
\mathrm{Grp}_{E_\infty}(\mathcal{S}) \times \mathrm{Grp}_{E_\infty}(\mathcal{S}) \arrow[ur, "- \boxtimes -"] &
\end{tikzcd}$$

This in particular implies that for any pointed space X we have

$$X \odot G \simeq FX \boxtimes G \simeq \Omega^\infty(\Sigma^\infty X \wedge B^\infty G)$$

where FX is the free E_∞-group on the pointed space X.

We can use compatiblity of tensors in $\mathrm{Grp}_{E_\infty}(\mathcal{S})$ to prove following splitting theorem for E_∞-groups.

Proposition 4.2 (R-Stonek-Valenzuela)**.** *Let G be a E_∞-group. Then for any pointed space X the cofiber sequence*

$$\begin{tikzcd} \{*\}_+ \arrow[r] & X_+ \arrow[l, dashed, bend left] \arrow[r] & X \end{tikzcd}$$

gives us an equivalence

$$X \otimes G \simeq G \times (X \odot G)$$

Remark 4.3. The induced map from $X \otimes G$ to G is is exactly $* \otimes G : X \otimes G \to * \otimes G \cong G$ and is a generalization of the composition map. In fact if $X = S^0$, then $* \otimes G = \mu : G \times G \to G$. So, we can think of this equivalence as a generalization of the shear map

$$(\mu, \pi_2) : G \times G \to G \times G$$

Thom Spectra and their Tensors

In order to compute THH of Thom spectra, we first have to define Thom spectra in the ∞-setting. The modern approach is primarily due to Ando, Blumberg, Gepner, Hopkins, Rezk and Antolín-Camarena, Barthel [ABGHR14, AB19].

How do we get a modern approach to Thom spectra? Let R be an E_∞-ring spectrum. The Thom construction assigns to each rank 1 (invertible) R-bundle $p : E \to B$ an R-module $M(p)$. The key realization is that we can classify each each bundle over a space X as a map $X \to BGL_1R$, as BGL_1 is the classifying space of rank 1 R-bundles. Thus we can think of the Thom construction as a functor

$$M : \mathcal{S}_{/BGL_1R} \to \mathrm{Mod}_R$$

and this functor preserves colimits. Using ∞-categorical machinery we know that any colimit preserving functor out of $\mathcal{S}_{/BGL_1R}$ is uniquely determined by its value on $* \to BGL_1R$. This map classifies the trivial bundle $R \to *$ and the Thom construction for the trivial bundle over the point is just the R-module R itself, which means the Thom construction restricted to BGL_1R is just the inclusion functor. This complete determines M as a left Kan extension

$$\begin{tikzcd} BGL_1R \arrow[r, hook] \arrow[d] & \mathrm{Mod}_R \\ \mathcal{S}_{/BGL_1R} \arrow[ur, dashed, "M"] & \end{tikzcd}$$

which we can describe explicitly as

$$M(f) = \mathrm{colim}(G \xrightarrow{f} BGL_1R \hookrightarrow \mathrm{Mod}_R).$$

We will use a slight generalization of this expression above.

Definition 5.1. Let R be a E_∞-ring spectrum. We define the *Picard space*, $\mathrm{Pic}(R)$, to be the underlying maximal ∞-groupoid of invertible R-modules. Notice Mod_R has a symmetric monoidal structure and thus $\mathrm{Pic}(R)$ is an E_∞-group.

Concretely $\mathrm{Pic}(R) \simeq \pi_0(R) \times BGL_1(R)$ and we can think of $BGL_1(R)$ as the connected component of the identity element of the E_∞-group $\mathrm{Pic}(R)$.

Definition 5.2. Let G be a group-like E_∞-space and $f : G \to \mathrm{Pic}(R)$ be a map of E_∞-spaces. We define the *Thom spectrum* of f, $M(f)$ as the following colimit

$$M(f) = \mathrm{colim}(G \xrightarrow{f} \mathrm{Pic}(R) \hookrightarrow \mathrm{Mod}_R).$$

1. Here we are thinking of the space G as an ∞-category in which every morphism is invertible, also called ∞-groupoid.
2. Although $M(f)$ takes values in Mod_R but it actually has the structure of a E_∞-R-algebra as long as the map $f : G \to \mathrm{Pic}(R)$ is E_∞. We can define $M(f)$ for more general maps, however, we want to focus on the case where the resulting object $M(f)$ is an E_∞-R-algebra.
3. $M(-)$ gives us a symmetric monoidal functor

$$M : \mathrm{Grp}_{E_\infty}(\mathcal{S})_{/\mathrm{Pic}(R)} \to \mathrm{CAlg}_R$$

This means that for any two E_∞ maps $f : G \to \mathrm{Pic}(R)$ and $g : H \to \mathrm{Pic}(R)$ we have an equivalence

$$M(G\times H \xrightarrow{f\times g} \mathrm{Pic}(R)\times\mathrm{Pic}(R) \xrightarrow{\mu} \mathrm{Pic}(R)) \simeq M(f : G \to \mathrm{Pic}(R))\wedge_R M(g : G \to \mathrm{Pic}(R))$$

So, in particular, if $f = g$ we get

$$Mf\wedge_R Mf \simeq M(G\times G \xrightarrow{f\times f} \mathrm{Pic}(R)\times\mathrm{Pic}(R) \xrightarrow{\mu} \mathrm{Pic}(R)) \simeq M(G\times G \xrightarrow{\mu} G \xrightarrow{f} \mathrm{Pic}(R))$$

Example 5.3. Let $f : G \to \mathrm{Pic}(R)$ be the trivial map, then $M(f) = \mathbb{S}[G] \wedge R$.

This last example suggests to us that we should think of a Thom spectrum has a "twisting" of a suspension spectrum. The more complicated the map f is the more we are twisting our suspension spectrum.

THH of Thom Spectra Revisited

Having defined $M(f)$, we now want to study $X \otimes M(f)$ for some space X.

Theorem 6.1 (R-Stonek-Valenzuela)**.** *Suppose $f : G \to \mathrm{Pic}(R)$ is an E_∞-map and X is pointed. There is an equivalence of E_∞-R-algebras.*

$$X \otimes_R Mf \simeq Mf \wedge \mathbb{S}[X \odot G].$$

Proof. Here is an idea of the proof. We get an equivalence of E_∞-R-algebras using the universal property of $M(f)$.

$$X \otimes_R Mf \simeq M(X \otimes G \xrightarrow{f(id_X\otimes *)} \mathrm{Pic}(R))$$

But we also have $X \otimes G \simeq G \times (X \odot G)$, so the above is equivalent to

$$M(G \times (X \odot G) \xrightarrow{f\pi_1} \mathrm{Pic}(R)) \simeq M(f) \wedge_R M(X \odot G \xrightarrow{*} \mathrm{Pic}(R)) \simeq$$

$$Mf \wedge_R (R \wedge \mathbb{S}[X \odot G]) \simeq Mf \wedge \mathbb{S}[X \odot G]$$

where in the first equivalence we used the fact that M is symmetric monoidal $\square$

In particular then we have following corollary.

Corollary 6.2. *Suppose G is an E_∞-group. Then there is an equivalence of E_∞-R-algebras*

$$THH^R(Mf) = S^1 \otimes_R Mf \simeq Mf \wedge \mathbb{S}[BG].$$

Here is another corollary of the proof.

Corollary 6.3. *Suppose G is an E_∞-group. Then there is an equivalence of E_∞-R-algebras*

$$Mf \wedge_R Mf \simeq S^0 \otimes Mf \simeq Mf \wedge \mathbb{S}[G].$$

This equivalence is commonly known as the *Thom isomorphism* [Ma79]. We can use this result as a guide to see why our result holds in the first place. The classical proof of the Thom isomorphism theorem goes as follows. We have following diagram of E_∞-spaces:

$$\begin{tikzcd} G \times G \arrow[rr, "g:(x,y)\mapsto(xy,y)", "\simeq"'] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\pi_1"] \\ & G & \end{tikzcd}$$

where the map at the top g is an equivalence (with inverse $(x, y) \mapsto (xy^{-1}, y)$. Thus we get an equivalence

$$M(g) : M(G \times G \xrightarrow{f\mu} \mathrm{Pic}(R)) \xrightarrow{\simeq} M(G \times G \xrightarrow{f\pi_1} \mathrm{Pic}(R))$$

which reduces to an equivalence $Mf \wedge_R Mf \simeq Mf \wedge_R (R \wedge \mathbb{S}[G])$.

The construction that we just presented makes use of the fact that the composition maps $G \times G \to G$ is simply the universal map of the coproduct diagram

$$\begin{tikzcd} G \arrow[r, "\iota_1"] \arrow[dr, "id_G"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\iota_2"'] \arrow[dl, "id_G"] \\ & G & \end{tikzcd}$$

The map $* \otimes id : X \otimes G \to G$ satisfies the exact same universal property for the diagram $X \xrightarrow{\{G\}} \mathrm{Grp}_{E_\infty}(\mathcal{S})$. So, we can think of it as gathering all the essential information at one specific point via a generalization of the product, from where we can use a projection map, which we can capture in a diagram.

$$\begin{tikzcd} X \otimes G \arrow[rr, "\simeq"] \arrow[dr, "*\otimes id_G"'] & & G \times (X \odot G) \arrow[dl, "\pi_1"] \\ & G & \end{tikzcd}$$

and we can literally recover the original diagram simply by using $X = S^0$.

Moving beyond Thom Spectra

Having results for Thom spectra is nice, but obviously not every spectrum is a Thom spectrum and so we would like to generalize our computation. One prominent examples is KU, we know that $KU \simeq \mathbb{S}[K(\mathbb{Z}, 2)](x^{-1})$, where $x \in \pi_2\mathbb{S}[K(\mathbb{Z}, 2)]$. Clearly $\mathbb{S}[K(\mathbb{Z}, 2)]$ is a Thom spectrum, so our next goal is to show that results are preserved after we invert an element.

Theorem 7.1 (R-Stonek-Valenzuela)**.** *Let X be a connected pointed space. Let R be an E_∞-ring spectrum and $x \in \pi_*(R)$. Then we have an equivalence*

$$(X \otimes R)[x^{-1}] \simeq (X \otimes R) \wedge_R R[x^{-1}] \xrightarrow{\simeq} X \otimes R[x^{-1}]$$

In fact this theorem is a special case of a more general theorem about étale maps of which maps of the form $R \to R[x^{-1}]$ are an important example. Combining this with previous results we get following.

Theorem 7.2. *Let $f : G \to \mathrm{Pic}(R)$ be a map of grouplike E_∞-spaces and $x \in \pi_*(Mf)$. Let X be a connected pointed space. Then*

$$X \otimes (Mf[x^{-1}]) \simeq Mf[x^{-1}] \wedge \mathbb{S}[X \odot G]$$

In particular we get.

Corollary 7.3.

$$X \otimes (\mathbb{S}[G][x^{-1}]) \simeq \mathbb{S}[G][x^{-1}] \wedge \mathbb{S}[X \odot G]$$

Example 7.4. We can now apply this result to KU to deduce that for any pointed connected space X we have

$$X \otimes KU \simeq KU \wedge \mathbb{S}[X \odot K(\mathbb{Z}, 2)]$$

So, in particular

$$THH(KU) \simeq KU \wedge \mathbb{S}[BK(\mathbb{Z}, 2)]$$

This last result was actually the starting point of this whole project. $THH(KU)$ was also computed by Bruno Stonek in his thesis using model categorical methods from [EKMM95].

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