Thom spectra, higher THH and Tensors in ∞-Categories

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A New Perspective on THH

The goal of this talk is to show how we can use presentable ∞-category theory to reformulate our understanding of THH.

1. Recall classical perspective THH and its motivation!
2. Why even a modern perspective?
3. Tensors of Presentable ∞-Categories
4. Thom Spectra
5. Computing THH
6. Proof?

THH: The Classical Story for Rings

Let A be a k-algebra. Then

$$\begin{tikzcd}
A & A \otimes_k A \arrow[l, shift left=2] \arrow[l, shift right=2] & A \otimes_k A \otimes_k A \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] \\
\arrow[from=1-1, to=1-2] \arrow[from=1-2, to=1-3, shift left=2] \arrow[from=1-2, to=1-3, shift right=2] \arrow[from=1-3, to=1-4, shift left=4] \arrow[from=1-3, to=1-4] \arrow[from=1-3, to=1-4, shift right=4]
\end{tikzcd}$$

with

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } i < n \\ a_n a_0 \otimes \ldots \otimes a_{n-1} & \text{if } i = n \end{cases}$$

and

$$s_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n$$

The *Hochschild homology* $HH(A)$ is the homology of this complex.

THH: The Classical Story for E_∞-Rings

Let A be an E_∞-R-algebra. Then we can define a simplicial E_∞-R-algebra

$$\begin{tikzcd}
A \arrow[r, <-, shift left=2] \arrow[r, ->] \arrow[r, <-, shift right=2] & A \wedge_R A \arrow[r, <-, shift left=4] \arrow[r, ->, shift left=2] \arrow[r, <-] \arrow[r, ->, shift right=2] \arrow[r, <-, shift right=4] & A \wedge_R A \wedge_R A \arrow[r, <-, shift left=6] \arrow[r, ->, shift left=4] \arrow[r, <-, shift left=2] \arrow[r, ->] \arrow[r, <-, shift right=2] \arrow[r, ->, shift right=4] \arrow[r, <-, shift right=6] & \cdots
\end{tikzcd}$$

We define the *topological Hochschild homology* E_∞-R-algebra $THH^R(A)$ as the geometric realization of this simplicial object.

First studied by Bökstedt (1985) and in this form Elmendorf-Kriz-Mandell-May (1996).

Why THH?

The key word is *algebraic K-theory*.

$$\begin{tikzcd}
K \arrow[r, "\text{Cyclotomic Trace}"] \arrow[dr, "\text{Dennis Trace}"'] & TC \arrow[d] \\
 & THH
\end{tikzcd}$$

The **cyclotomic trace** preserves information (Goodwillie-Dundas-McCarthy 2012).

Why a new Approach?

Theorem (Schlichtkrull, 2011)

Let $f : G \to BGl_1(\mathbb{S})$ *be a map of* E_∞*-groups with* $M(f)$ *the associated Thom spectrum. Then we have an equivalence of* E_∞*-rings*

$$THH(M(f)) \simeq Mf \wedge \mathbb{S}[BG].$$

- $\mathbb{S}[BG] = \Sigma^\infty BG$ with ring structure

A New Approach Can Avoid these Difficulties

" ... However, when trying to make this argument precise one encounters several technical difficulties. First of all one needs A and $T(f)$ to be **cofibrant** *in order to control the homotopy type of the Loday functors but unfortunately the Thom S-algebra associated to a* **cofibrant** *$\mathcal{I}$-space A need not be* **cofibrant***. It is also not clear that the* **T-goodness condition** *for an object in $\mathcal{IU}/BF$ is preserved under* **cofibrant replacement***. (The latter difficulty is caused by the technical subtlety that whereas* **Hurewicz cofibrations** *are preserved under pullback along* **Hurewicz fibrations***, the behavior under pullback along* **Serre fibrations** *is not well understood)... "*

- Christian Schlichtkrull

Hence, we need an ∞-Categorical Approach

So, the goal is to generalize the constructions to the ∞-categorical setting and realize the dream of Schlichtkrull, while avoiding all the model categorical pitfalls!

A Good Start

Theorem (McClure-Schwänzl-Vogl, 1997)

There is a tensor E_∞-ring $K \otimes R$ for every space K and E_∞-ring R and we have an equivalence of E_∞-rings

$$\mathrm{THH}(R) \simeq S^1 \otimes R$$

What is an ∞-Category?

The technical term here is *quasi-category* $\mathcal{C}$. If that is not familiar, then just think of the following data:

1. We have objects X, Y, ... in $\mathcal{C}$.
2. We have a mapping **space** $\mathrm{Map}_{\mathcal{C}}(X, Y)$
3. All classical categorical terms (limits, adjunctions, presentability, ...) still hold, although some need to be adjusted.

Most work here goes back to Joyal and Lurie.

Presentable Categories vs. Presentable ∞-Categories

Definition

A category $\mathcal{P}$ is called *locally presentable* if there exists a **small** category $\mathcal{C}$ and an adjunction

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \arrow[r, shift left=1.5, "L"] \arrow[r, phantom, "\perp"] & \mathcal{P} \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

such that i is fully faithful .

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such that i is fully faithful and L is accessible.

Presentable ∞-Categories and Tensors

Slogan!

There is an analogy between symmetric monoidal categories (Gepner-Groth-Nikolaus, 2016):

$$\begin{tikzcd}
 & (\text{Abelian Groups}, \otimes) & \\
(\text{Locally Presentable Categories}, \otimes) \arrow[ur, Leftrightarrow] \arrow[rr, Leftrightarrow] & & (\text{Presentable } \infty\text{-Categories}, \otimes) \arrow[ul, Leftrightarrow]
\end{tikzcd}$$

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Canonical Module	A $\mathbb{Q}$-vector space $\leftrightarrow$ $\begin{tikzcd} \mathbb{Z} \times A \arrow[r] \arrow[d] & A \\ \mathbb{Q} \times A \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$	$\mathcal{C}$ is pointed $\leftrightarrow$ $\begin{tikzcd} \mathcal{S}et \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{S}et_* \times \mathcal{C} \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$	$\mathcal{C}$ is stable $\leftrightarrow$ $\begin{tikzcd} \mathcal{S} \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathrm{Sp} \times \mathcal{C} \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$

Tensors of E_∞-Groups

Example

$\mathrm{Grp}_{E_\infty}(\mathcal{S})$ is the unit of **additive presentable ∞-categories**.

$$\begin{tikzcd}
- : & \mathcal{S} \times \mathrm{Grp}_{E_\infty}(\mathcal{S}) \arrow[dr, "- \otimes -"] & \\
& & \mathrm{Grp}_{E_\infty}(\mathcal{S})
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\end{tikzcd}$$

Old Notation:

$$\underline{X \odot G} \simeq \mathrm{Free}(X) \boxtimes G \simeq \underline{\Omega^\infty(\Sigma^\infty X \wedge B^\infty G)}$$

Splitting Groups

Let X be a pointed space

$$+ \longmapsto x_0$$

$$\begin{tikzcd}
\{*\}_+ \arrow[r] & X_+ \arrow[r] \arrow[l, dashed, bend left] & X
\end{tikzcd}$$

$$* \longmapsto x_0$$

$$\begin{tikzcd}
S^0 = \{*\}_+ \odot G \arrow[r] & \underline{X_+ \odot G} \arrow[r] \arrow[l, dashed, bend left] & X \odot G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r] & X \otimes G \arrow[r] \arrow[l, dashed, bend left] & X \odot G \, .
\end{tikzcd}$$

Splitting Groups

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\{*\}_+ \arrow[r] & X_+ \arrow[r] \arrow[l, dashed, bend left] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\}_+ \odot G \arrow[r] & X_+ \odot G \arrow[r] \arrow[l, dashed, bend left] & X \odot G
\end{tikzcd}$$

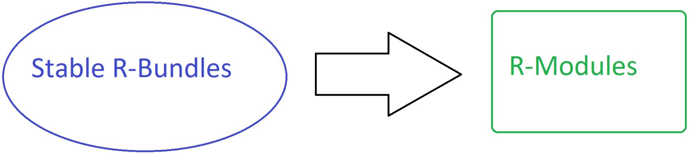
$$\begin{tikzcd}
G \arrow[r] & X \otimes G \arrow[r] \arrow[l, dashed, bend left] & X \odot G\ .
\end{tikzcd}$$

This gives us an equivalence:

$$X \otimes G \simeq G \times (X \odot G).$$

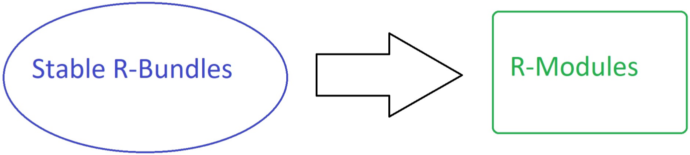
What is a Thom Spectrum?

R: E_∞-ring $\qquad GL_1 R = \mathrm{Aut}(R, R)$ inside E_∞ rings



$$M : \quad \mathcal{S}_{/BGL_1 R} \longrightarrow \mathrm{Mod}_R$$

What is a Thom Spectrum?



$$M : \quad \mathcal{S}_{/BGL_1R} \longrightarrow \mathrm{Mod}_R$$

1. $M(Triv \to *) = M(* \to BGL_1R) = R$
2. M is colimit preserving.

(Ando-Blumberg-Gepner-Hopkins-Rezk, 2014).

A New Perspective on Thom Spectra I

- G: E_∞-group
- R: E_∞-ring spectrum

Left Kan extension:

$$\begin{tikzcd}
\{*\}_{/BGL_1R} \simeq BGL_1R \arrow[r, hook, "\{R\}"] \arrow[d, hook] & \mathrm{Mod}_R \\
\mathcal{S}_{/BGL_1R} \arrow[ur, dashed, "M"] &
\end{tikzcd}$$

$$M(f : X \to BGL_1R) = \mathrm{colim}(X \to BGL_1R \hookrightarrow \mathrm{Mod}_R)$$

A New Perspective on Thom Spectra II

Definition

An R-module M is *invertible* if there exists an R-module S, such that $M \wedge_R S \simeq R$. Let $\mathrm{Pic}(R)$ be the subgroupoid of *invertible R-modules* in Mod_R.

We can extend the Thom spectrum to $\mathrm{Pic}(R)$.

R inv

$\Sigma^n R$ inv

$\Sigma^{-n} R$

$$\begin{tikzcd}
\mathrm{Pic}(R) \arrow[r, hook] \arrow[d, hook] & \mathrm{Mod}_R \\
\mathcal{S}_{/\mathrm{Pic}(R)} \arrow[ur, dashed, "M"] &
\end{tikzcd}$$

$$\boxed{M(f : X \to \mathrm{Pic}(R)) = \mathrm{colim}(X \to \mathrm{Pic}(R) \hookrightarrow \mathrm{Mod}_R)}$$

Properties of Thom Spectra

1. The construction is symmetric monoidal:

$$M : \mathrm{Grp}_{E_\infty}(\mathcal{S})_{/\mathrm{Pic}(R)} \to \mathrm{CAlg}_R$$

2. In particular

$$Mf \wedge_R Mf \simeq M(G \times G \xrightarrow{\mu} \underline{G \xrightarrow{f}} \mathrm{Pic}(R))$$

3. For example, if $f : G \to \mathrm{Pic}(R)$ is the trivial map, then $M(f) = \mathbb{S}[G] \wedge R$. $\searrow * \nearrow$

$M(*) = R$

Tensor of Thom Spectra

We finally have all the background to do some computations!

Tensor of Thom Spectra

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Theorem (R-Stonek-Valenzuela)

Suppose $f : G \to \mathrm{Pic}(R)$ is an E_∞-map and X is pointed. There is an equivalence of E_∞-R-algebras.

$$X \otimes_R Mf \simeq Mf \wedge \mathbb{S}[X \odot G].$$

Interesting Implications I

1. THH:

$S^1 \odot G \simeq BG$

$$\mathrm{THH}(Mf) \simeq Mf \wedge \mathbb{S}[S^1 \odot G] \simeq Mf \wedge \mathbb{S}[BG]$$

↑ Schlichtkrull

2. Thom Isomorphism:

$BG \odot S^0 = BG$

$$Mf \wedge_R Mf \simeq S^0 \underset{R}{\otimes} Mf \simeq Mf \wedge \mathbb{S}[BG]$$

3. Extends the result by Schlichtkrull to non-connective Thom spectra.

Interesting Implications II

Theorem (R-Stonek-Valenzuela)

Let $f : G \to \mathrm{Pic}(R)$ be a map of grouplike E_∞-spaces and $x \in \pi_*(Mf)$. Let X be a connected pointed space. Then

$$X \otimes (Mf[x^{-1}]) \simeq Mf[x^{-1}] \wedge \mathbb{S}[X \odot G]$$

Interesting Implications II

Theorem (R-Stonek-Valenzuela)

Let $f : G \to \mathrm{Pic}(R)$ *be a map of grouplike* E_∞*-spaces and* $x \in \pi_*(Mf)$*. Let* X *be a connected pointed space. Then*

$$X \otimes (Mf[x^{-1}]) \simeq Mf[x^{-1}] \wedge \mathbb{S}[X \odot G]$$

Example

For any pointed connected space X: $KU = \underbrace{\mathbb{S}[K(\mathbb{Z},2)]}_{\text{Thom spectrum}}[\beta^{-1}]$

$$X \otimes KU \simeq X \otimes \mathbb{S}[K(\mathbb{Z},2)][\beta^{-1}] \simeq KU \wedge \mathbb{S}[X \odot K(\mathbb{Z},2)]$$

$$\underline{THH(KU) \simeq KU \wedge \mathbb{S}[BK(\mathbb{Z},2)]}$$

Proven originally in a model category setting by Stonek.

Proof I

$$X \otimes_R Mf \simeq Mf \wedge S[X \odot G]$$

$$X \otimes_R Mf \simeq M\big(X \otimes G \xrightarrow[r \otimes G]{} G \xrightarrow{f} Pic(R)\big)$$

$\uparrow$ colimit

$$\simeq M\big(G \times X \odot G \xrightarrow[\pi_1]{} G \xrightarrow{f} Pic(R)\big)$$

$$\simeq M\big(G \xrightarrow{f} Pic(R)\big) \wedge_R M\big(X \odot G \xrightarrow{\{R\}} Pic(R)\big)$$

$$\simeq Mf \wedge_{\cancel{R}} \cancel{R} \wedge S[X \odot G]$$

$$\simeq Mf \wedge S[X \odot G]$$

Proof II

$$X = S^0$$

$$S^0 \otimes_R Mf \simeq Mf \wedge S[S^0 \otimes G]$$

$$Mf \wedge_R Mf \simeq Mf \wedge S[G]$$

$$\begin{tikzcd}
G \times G \arrow[rr, "{(x,y) \mapsto (xy, y^{-1})}", "\simeq"'] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\pi_1"] \\
& G \arrow[d, "f"] & \\
& Pic(R) &
\end{tikzcd}
\longrightarrow
M\left(\begin{array}{c} G \times G \\ \downarrow \mu \\ G \\ \downarrow \\ Pic(R) \end{array}\right) \xrightarrow{\simeq} M\left(\begin{array}{c} G \times G \\ \downarrow \pi_1 \\ G \\ \downarrow \\ Pic(R) \end{array}\right)$$

$$Mf \wedge_R Mf \longrightarrow Mf \wedge S[G]$$

Proof III

This argument generalizes

$$\begin{tikzcd}
X \otimes G \arrow[rr, "\simeq"] \arrow[dr] & & \boxed{G} \times (X \odot G) \arrow[dl] \\
& G \arrow[d] & \\
& \mathrm{Pic}(R) &
\end{tikzcd}
\rightsquigarrow
M\left(\begin{tikzcd} X \otimes G \arrow[d] \\ G \arrow[d] \\ \mathrm{Pic}(R) \end{tikzcd}\right)
\xrightarrow{\simeq}
M\left(\begin{tikzcd} G \times (X \odot G) \arrow[d] \\ G \arrow[d] \\ \mathrm{Pic}(R) \end{tikzcd}\right)$$

$$X \otimes Mf \xrightarrow{\simeq} Mf \wedge \mathbb{S}[X \odot G]$$

The End!

For more details see:

- Thom spectra, higher THH and tensors in ∞-categories
- Nima Rasekh, Bruno Stonek, Gabriel Valenzuela
- arXiv:1911.04345

Thank you!

Questions?