An Introduction to TFTs

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Topological field theories arise when we want to assign algebraic invariants to manifolds in an cobordism invariant way. In order to make sense of that we will discuss following topics

1. Cobordism and Cobordims invariants
2. Topological Field Theories (TFTs) and some basic facts about TFTs
3. Low Dimensional Examples
4. Extended Locality

I should note all of this material comes from talks that Dan Freed gave in the conference "Topological and Geometric Methods in QFT" in Montana State University ([Fr17]) and notes by by Carqueville and Runkel ([CR17]).

Cobordism

A bordism can be thought of as a "homotopy of manifolds" and was initially introduced by Poincare.

Definition 1.1. Let M_1 and M_2 be two n-dimensional manifolds. A *bordism* from M_1 to M_2 is a tuple $(W, p, \varphi_1, \varphi_2)$, where W is an $n+1$-dimensional manifold such that $\partial W = M_1 \coprod M_2$, $p : \partial W \to \{0, 1\}$ is a map and φ_1 and φ_2 are embeddings

$$\varphi_1 : [0, 1) \times M_1 \to W$$

$$\varphi_2 : (-1, 0] \times M_2 \to W$$

This definition gives us an equivalence relation.

Definition 1.2. We say two n dimensional manifolds M_1, M_2 are *bordant* if there is a a bordism between them.

Definition 1.3. Let Ω_d be the set of bordism equivalence classes of d-dimensional manifolds.

Turns out this set is nice. It's an abelian group.

Proposition 1.4. *Ω_d is an abelian group.*

Proof. The operation is disjoint union. The identity is the empty manifold. The inverse of each manifold is itself. □

Remark 1.5. We can study bordisms of manifolds that have extra structure. In particular, we can talk about oriented bordisms between oriented manifolds. The resulting group is denoted by $\Omega_d(SO)$. We can do similar things for spin manifolds or manifolds with a complex structure.

Having all of this we can finally define *bordism invariants*

Definition 1.6. A *classical bordism invariant* is a group homomorphism from Ω_d to $\mathbb{Z}$.

In other words it assign algebraic data to each bordism class. There are various interesting examples of such.

Example 1.7.

1. *Signature*: An invariant of a $4d$ dimensional oriented manifolds

$$\Omega_{4d}(SO) \to \mathbb{Z}$$

2. *$\hat{A}$-genus*: An invariant of $2d$ dimensional spin manifolds

$$\Omega_{2d}(Spin) \to \mathbb{Z}$$

3. *Todd Genus*: An invariant of manifolds with complex structure.

$$Todd : \Omega_{2k}(U) \to \mathbb{Z}$$

Remark 1.8. Some interesting algebraic invariants do not respect bordisms. An important example of that is the Euler characteristic. It only works if we think of it as a map to $\mathbb{Z}/2$.

This is all nice and good, but there is something missing. The algebraic invariant cannot see **how** two manifolds are bordant, but just the fact that they are. All the equivalence information is inevitably lost. Thus the goal is to add more algberaic data to be able to preserve that information. This leads to the concept of *categorification.*

Topological Field Theories (TFTs)

In order to be able to preserve the information of equivalences we need to generalize our constructions from sets to categories. This leads to following definition:

Definition 2.1. A *bordism category $Bord_{n,n-1}$* is the symmetric monoidal category defined as follows

Obj Objects are $n-1$ dimensional manifolds

Mor Morphisms $Hom(M_1, M_2)$ are diffeomorphism classes of bordisms from M_1 to M_2. In particular, composition is by gluing bordisms and the identity is $[0,1] \times M$

SMon The symmetric monoidal product is disjoint union. The identity is the empty manifold.

This is the proper categorification of Ω_d. As before, we can adjust the category if we want to study manifolds with structure. Now what is the categorification of the integers? Turns out it is *Vector spaces.* We will take complex vector spaces here. This category is symmetric monoidal with the tensor product.

Definition 2.2. A *topological field theory* (TFT) is a symmetric monoidal functor

$$F : Bord_{n,n-1} \to Vect_{\mathbb{C}}$$

What can we say about this TFTs just by looking at the definition?

(1) We have $F(\emptyset) = \mathbb{C}$ as a symmetric monoidal functors preserve the unit. In particular,

$$Hom(\emptyset, \emptyset) = \text{Diffeomorphism classes of closed } n\text{-dimensional manifolds}$$

is mapped to $Hom(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. Thus, it assigns a complex number to each diffeomorphism class of closed n-dimensional manifolds.

(2) Every object in $Bord_{n,n-1}$ is *dualizable.*

Definition 2.3. An object M is dualizable if there is an object (dual) $\bar{M}$ and maps

$$u_M : \emptyset \to M \coprod \bar{M}$$

$$c_M : M \coprod \bar{M} \to \emptyset$$

such that c_m and u_M satisfy some reasonable commutativity conditions.

This is clearly true in $Bord_{n,n-1}$ as we can see from the picture that every manifold is a dual to itself. This means that $F(M)$ is dualizable as well. But in the world of vector spaces a dualizable objects are exactly the finite dimensional vector spaces. Thus every TFT actually maps into finite dimensional vector spaces.

Now that we have a working definition. Let's see some examples.

Low-Dimensional Examples

Let us see what we get if we look at low n. Here we will focus on TFTs on orientable manifolds.

(Dimension 1): Every oriented 0 manifold is a disjoint union of points each of which has a given orientation. Thus if I know where I map one positively oriented point, then I know what happens to everything else. The negative oriented point simply maps to the dual and the disjoint unions to the tensor product.

Moreover, there are no relations that prevent us from picking any finite dimensional vector space we like. Thus we get

$$1, 0 \text{ oriented TFTs} = \text{finite dimensional vector spaces.}$$

That wasn't that exciting so let's move on.

(Dimension 2): Every oriented 1 manifold is a disjoint union of circles. Thus we might expect a result similar to the one above, however, it turns out this is not true. Because of the added dimension there are now relations which did not exist before.

The algebraic object we need is a *Frobenius algebra*.

Definition 3.1. A Frobenius algebra is a a vector space A such that

1. it has a associative unital algebra structure
2. it has a coassociative counital coalgebra structure
3. these two structures interace well with each other.

Having this definition the result should be the following

Theorem 3.2. *There is an equivalence between* 2 *dimensional TFTs and commutative Frobenius algebras.*

For the relevant pictures see Page 27 of [CR17].

Extended Locality: How Higher Categories enter the Picture

Field theories can be used to study manifolds. However the higher dimensional the manifold is, the harder we can study it using field theories. We would rather prefer to have more information.

Let me give one concrete example where this occurs.

We already mentioned that in $Bord_{n,n-1}$ we get a complex number $F(X)$ for each n-dimensional manifold X. If we cut that manifold X along a $n-1$ dimensional sub manifold Y, we get two pieces X_1, X_2. Which gives us two bordisms, from $\emptyset$ to Y and from Y to $\emptyset$. This means we get two maps

$$F(X_1) : F(Y) \to \mathbb{C}$$

$$F(X_2) : \mathbb{C} \to F(Y)$$

If we fix a basis $e_1, ..., e_n$ for $F(Y)$ and a dual basis $f_1, ..., f_n$ then we can write those maps explicitly as

$$F(X_1) = \sum a_i e_i$$

$$F(X_1) = \sum b_i f_i$$

and as

$$F(X) = F(X_1) \circ F(X_2)$$

we must have

$$F(X) = \sum a_i b_i$$

That way we can compute higher dimensional information using lower dimensional pieces and gluing them.

We want to do something similar for an $n-1$ dimensional manifold M. However, the previous approach doesn't exactly work. First of all the outcome has to be a vector space and not a number. If we just tried, then first we would cut M along a $n-2$ dimensional manifold N to get two pieces M_1 and M_2. By analogy we get maps

$$F(M_1) = \oplus V_i c_i$$

$$F(M_2) = \oplus W_i d_i$$

where V_i and W_i are now vector spaces. When we glue things then we should get

$$F(M) = \oplus V_i \otimes W_i$$

But here is the problem. What are those c_i and d_i? They should be invariants of the $n-2$ dimensional manifold N we cut along. But our field theory cannot see $n-2$ dimensional manifolds.

The correct approach then is to define a new *extended* field theory $Bord_{n,n-1,n-2}$. That has objects $n-2$-dimensional manifolds, maps bordisms of $n-2$-dimensional manifolds and then 2-maps diffeomorphism classes of bordism of bordisms of $n-2$-dimensional manifolds. This object cannot be studied by just using a category anymore as the mappings have far more information than a simple set. For that reason we need to resort to higher categorical ideas.

References

[CR17] N. Carqueville, I. Runkel. *Introductory lectures on topological quantum field theory.* arXiv preprint arXiv:1705.05734 (2017).

[Fr17] D. Freed. *Topological and Geometric Methods in QFT*, Conference Lecture series at Montana State University