THH and Shadows of Bicategories

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What's this talk about?

- Goal of this talk is to understand connection between **topological Hochschild homology** and **shadows**.
- In order to do that we review the relevant concepts.
- If time permits we will look applications to Morita invariance and possible further applications.
- For more details see the paper: *Shadows are Bicategorical Traces* (arXiv:2109.02144).

Hochschild Homology

Let A be an associative unital algebra over a ring k and M an (A, A)-bimodule. Then Hochschild homology $HH_\bullet(M, A)$ is defined as the homology of the complex

$$M \leftleftarrows M \otimes_k A \;\substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow}\; M \otimes_k A \otimes_k A \;\substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow}\; \cdots$$

with

$$d_i(m \otimes ... \otimes a_{n-1} \otimes a_n) = \begin{cases} m \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n & \text{if } i < n \\ a_n m \otimes ... \otimes a_{n-1} & \text{if } i = n \end{cases}$$

and

$$s_i(m \otimes ... \otimes a_n) = m \otimes ... \otimes a_i \otimes 1 \otimes a_{i+1} \otimes ... \otimes a_n$$

Simplicial Abelian Groups are Complexes

We can translate the simplicial abelian group to a complex by adding with alternating signs:

$$M \xleftarrow{a_0 m - m a_0} M \otimes_k A \xleftarrow{(ma_0, a_1) - (m, a_0 a_1) + (a_1 m, a_0)} M \otimes_k A \otimes_k A \longleftarrow \ldots$$

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In particular:

$$HH_0(M, A) = M/[M, A]$$

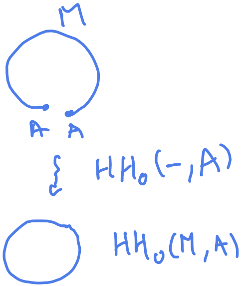
Hochschild Homology vs. Tensor

$${}_A M_B \quad {}_B N_C \rightsquigarrow {}_A M \otimes_B N_C \text{ "=" } M \times N/\sim$$

$$(mb,n) \sim (m,bn)$$

$${}_A M_A \rightsquigarrow HH_0(M,A) = M/\sim$$

$$am \sim ma$$



Why Hochschild Homology in Homotopy Theory?

- If A is commutative then Hochschild homology recovers Kähler differentials of the corresponding variety and hence Hochschild homology has been used as a generalization to the non-commutative setting (Connes).

Why Hochschild Homology in Homotopy Theory?

- If *A* is commutative then Hochschild homology recovers Kähler differentials of the corresponding variety and hence Hochschild homology has been used as a generalization to the non-commutative setting (Connes).
- It became an object of interest in homotopy theory via the **Dennis trace map** from **algebraic K-theory**.
- This motivated a homotopical generalization of Hochschild homology to *topological Hochschild homology* with a generalization of the trace, by Bökstedt and EKMM.

Topological Hochschild Homology

Let A be a ring spectrum over k and M an (A, A)-bimodule over A. Then topological Hochschild homology $\mathrm{THH}(M, A)$ is defined as the spectrum

$$\mathrm{THH}(M, A) =$$

$$\left| M \leftleftarrows M \wedge_k A \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; M \wedge_k A \wedge_k A \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; \cdots \right|$$

with

$$d_i(m \otimes \ldots \otimes a_{n-1} \otimes a_n) = \begin{cases} m \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } i < n \\ a_n m \otimes \ldots \otimes a_{n-1} & \text{if } i = n \end{cases}$$

and

$$s_i(m \otimes \ldots \otimes a_n) = m \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n$$

From Rings to Categories

Algebraic K-theory of a ring is computed via its category of modules and so the input was generalized to categories that share certain features (such as *Waldhausen categories*).

Hence, THH has also been generalized to various categories and in particular here we focus on the case of *spectrally enriched categories*: meaning a category $\mathcal{C}$ with objects $X, Y, Z, \ldots$ and for two objects a mapping spectrum $\mathcal{C}(X, Y)$ and composition

$$\mathcal{C}(X, Y) \wedge \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z).$$

"Ring spectrum $=$ Spectrally enriched category with one object"

THH of Bimodules

Let $\mathcal{C}, \mathcal{D}$ be spectrally enriched categories. A $(\mathcal{C}, \mathcal{D})$-bimodule $\mathcal{M}$ is a spectrum valued functor

$$\mathcal{M} : \mathcal{C}^{op} \wedge \mathcal{D} \to \mathrm{Sp}$$

and for a $(\mathcal{C}, \mathcal{C})$-bimodule $\mathcal{M}$, $\mathrm{THH}(\mathcal{M}, \mathcal{C})$ is defined as

$$\mathrm{THH}(\mathcal{M}, \mathcal{C}) =$$

$$|\bigvee_{c_0,\ldots,c_n} \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_1, c_2) \wedge \ldots \wedge \mathcal{C}(c_{n-1}, c_n) \wedge \mathcal{M}(c_n, c_0)|$$

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$$\mathrm{THH}(\mathcal{M}, \mathcal{C}) = \left| \bigvee_{c_0, \ldots, c_n} \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_1, c_2) \wedge \ldots \wedge \mathcal{C}(c_{n-1}, c_n) \wedge \mathcal{M}(c_n, c_0) \right|$$

$\in \mathrm{Sp}$

Moreover, for two modules $\mathcal{M} : \mathcal{C}^{op} \wedge \mathcal{D} \to \mathrm{Sp}$, $\mathcal{N} : \mathcal{D}^{op} \wedge \mathcal{C} \to \mathrm{Sp}$ we can define the $(\mathcal{C}, \mathcal{C})$-module

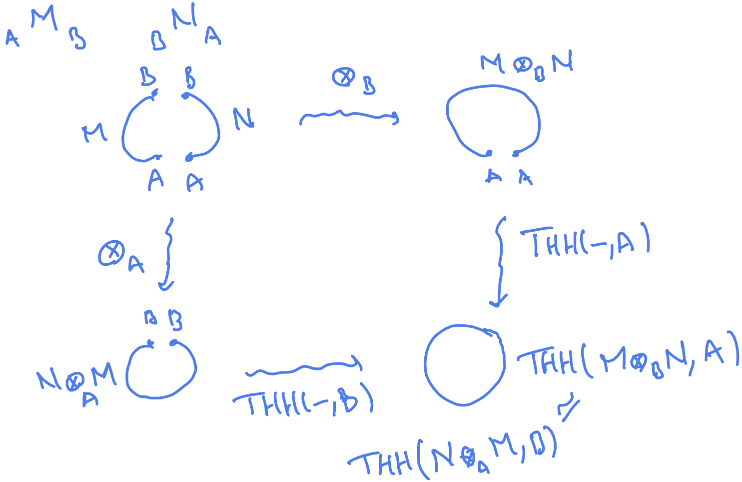
$$\mathcal{M} \otimes \mathcal{N} : \mathcal{C}^{op} \wedge \mathcal{C} \to \mathrm{Sp}$$

coend bar construction

as $\mathcal{M} \otimes \mathcal{N}(c, c') = \mathcal{M}(c, -) \wedge_{\mathcal{D}} \mathcal{N}(-, c')$ and similarly $\mathcal{N} \otimes \mathcal{M}$ and we now have the following interesting result:

$$\mathrm{THH}(\mathcal{M} \otimes \mathcal{N}, \mathcal{C}) \simeq \mathrm{THH}(\mathcal{N} \otimes \mathcal{M}, \mathcal{D}).$$

What does this mean?



Bicategories

A bicategory has the data:

- **Objects:** $X, Y, Z \in \mathrm{Obj}_{\mathcal{B}}$
- **Morphisms:** For two objects X, Y a category $\mathcal{B}(X, Y)$.
- **Composition:** For three objects X, Y, Z a composition $\mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z)$.
- **Identity:** For object X a unit morphism U_X which is an object in $\mathcal{B}(X, X)$.
- **Associator:** For three composable morphisms f, g, h a natural isomorphism
$$(hg)f \overset{a}{\cong} h(gf)$$
known as the *associator*.
- **Unitor:** For a morphisms, $f : X \to Y$, natural isomorphisms
$$f(U_X) \overset{r}{\cong} f \overset{l}{\cong} U_Y f$$
known as the right and left unitor.

Enter Shadows I

Dealing with this structure becomes increasingly challenging, motivating Ponto to introduce the notion of a **shadow**, which precisely axiomatized the key aspects of $\mathrm{THH}(\mathcal{M}, \mathcal{C})$.

Definition (Ponto)

Let $\mathcal{B}$ be a bicategory and $\mathcal{D}$ a category. A *shadow* on $\mathcal{B}$ with values in $\mathcal{D}$ is a functor

$$\langle\langle - \rangle\rangle : \coprod_{X \in \mathrm{Obj}(\mathcal{B})} \mathcal{B}(X, X) \to \mathcal{D}$$

such that for every pair of 1-morphisms $F : X \to Y$ and $G : Y \to X$ in $\mathcal{B}$, there is a natural isomorphism

$$\theta : \langle\langle FG \rangle\rangle \xrightarrow{\cong} \langle\langle GF \rangle\rangle.$$

Enter Shadows II

Moreover, for all $F : X \to Y$, $G : Y \to Z$, $H : Z \to X$, and $K : X \to X$ the following diagrams in $\mathcal{D}$ commute:

$$\begin{array}{ccccc}
\langle\langle H(GF)\rangle\rangle & \xrightarrow{\theta} & \langle\langle (GF)H\rangle\rangle & \xrightarrow{\langle\langle a\rangle\rangle} & \langle\langle G(FH)\rangle\rangle \\
\Big\downarrow{\scriptstyle \langle\langle a\rangle\rangle} & & & & \Big\uparrow{\scriptstyle \theta} \\
\langle\langle (HG)F\rangle\rangle & \xrightarrow{\theta} & \langle\langle F(HG)\rangle\rangle & \xrightarrow{\langle\langle a\rangle\rangle} & \langle\langle (FH)G\rangle\rangle
\end{array}$$

$$\begin{array}{ccccc}
\langle\langle KU_X\rangle\rangle & \xrightarrow{\theta} & \langle\langle U_X K\rangle\rangle & \xrightarrow{\theta} & \langle\langle KU_X\rangle\rangle \\
& {\scriptstyle \langle\langle r\rangle\rangle}\searrow & \Big\downarrow{\scriptstyle \langle\langle l\rangle\rangle} & \swarrow{\scriptstyle \langle\langle r\rangle\rangle} & \\
& & \langle\langle K\rangle\rangle & &
\end{array}$$

THH is a Shadow

Let Mod be the bicategory with objects spectrally enriched categories and morphisms $\mathrm{Mod}(\mathcal{C}, \mathcal{D}) = \mathrm{Mod}_{(\mathcal{C},\mathcal{D})}$, the homotopy category of $(\mathcal{C}, \mathcal{D})$-bimodules.

Theorem (Campbell-Ponto)

The functor

$$\mathrm{THH} : \coprod_{\mathcal{C}} \mathrm{Mod}_{(\mathcal{C},\mathcal{C})} \to \mathrm{Sp}$$

is a shadow on Mod *with values in* Sp.

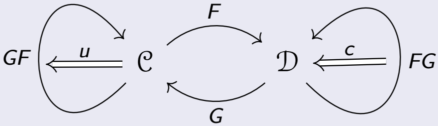
Notice the key input is the equivalence

$$\mathrm{THH}(\mathcal{M} \otimes \mathcal{N}, \mathcal{C}) \simeq \mathrm{THH}(\mathcal{N} \otimes \mathcal{M}, \mathcal{D}).$$

Morita Equivalence

Definition

Let $\mathcal{B}$ be a bicategory. We say two morphisms $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are *Morita dual* if there is a diagram of the form



satisfying the triangle identities and *Morita equivalent* if u, c are isomorphisms.

Shadows are Morita Invariant

Proposition (Campbell-Ponto)

Shadows are Morita invariant, meaning if we have a shadow $\langle\langle - \rangle\rangle$ *on* $\mathcal{B}$ *and* F *and* G *are Morita equivalent, then*

$$\langle\langle \mathrm{id}_{\mathcal{C}} \rangle\rangle \cong \langle\langle \mathrm{id}_{\mathcal{D}} \rangle\rangle$$

given via

$$\langle\langle \mathrm{id}_{\mathcal{C}} \rangle\rangle \overset{\langle\langle u \rangle\rangle}{\to} \langle\langle GF \rangle\rangle \overset{\theta}{\cong} \langle\langle FG \rangle\rangle \overset{\langle\langle c \rangle\rangle}{\to} \langle\langle \mathrm{id}_{\mathcal{D}} \rangle\rangle$$

This in particular generalizes Morita invariance of THH.

Question?

Any questions about THH or shadows?

Why Homotopy Coherence?

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2. There are now ∞-categorical approaches to THH, due to Nikolaus-Scholze, Berman, Ayala-Francis, ... Shadows could only be used to axiomatize them at the level of homotopy categories and a proper axiomatization would require an $(\infty, 2)$-categorical version of a shadow.

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3. More fundamentally, there are coalgebraic version of THH, **topological coHochschild homology**, due to Shipley and Hess, and those cannot be studied at all at the point-set level (Péroux-Shipley).

THH of Bicategories

THH : Bicategory ⟶ Cat

Using ideas regarding THH of enriched ∞-categories due to Berman, we can define THH of a bicategory $\mathcal{B}$ as follows:

$$\mathrm{THH}(\mathcal{B}) \simeq$$

$$\Big| \coprod_{X_0,\dots,X_n} \mathcal{B}(X_0,X_1) \times \dots \times \mathcal{B}(X_n,X_0) \Big| \overset{\text{def}}{=} \Big| \coprod_{X_0,\dots,X_n} \mathcal{B}(X_0,\dots,X_n,X_0) \Big| \simeq$$

$$\Big| \coprod_{X_0} \mathcal{B}(X_0,X_0) \overset{d_0}{\underset{d_1}{\leftleftarrows}} \coprod_{X_0,X_1} \mathcal{B}(X_0,X_1,X_0) \overset{d_0}{\underset{d_2}{\leftleftarrows}} \coprod_{X_0,X_1,X_2} \mathcal{B}(X_0,X_1,X_2,X_0) \Big|$$

evaluated in the $(2,1)$-category of categories (pseudo-colimit).

A functor $\mathrm{THH}(\mathcal{B}) \to \mathcal{D}$ is called a **bicategorical trace**.

Category of Shadows

Definition

Let $\mathcal{B}$ be a bicategory and $\mathcal{D}$ a category, define the category $\mathcal{S}\mathrm{ha}(\mathcal{B}, \mathcal{D})$ as the category with objects shadows and morphisms natural transformations

$$\alpha : \langle\langle - \rangle\rangle_1 \to \langle\langle - \rangle\rangle_2$$

that commutes with the associator, unitor and the following diagram:

$$\begin{array}{ccc} \langle\langle FG \rangle\rangle_1 & \xrightarrow{\theta_1} & \langle\langle GF \rangle\rangle_1 \\ \downarrow{\scriptstyle \alpha_{FG}} & & \downarrow{\scriptstyle \alpha_{GF}} \\ \langle\langle FG \rangle\rangle_2 & \xrightarrow{\theta_2} & \langle\langle GF \rangle\rangle_2, \end{array} .$$

Shadows vs. THH

We now have the following main theorem.

Theorem (Hess-R.)

Let $\mathcal{B}$ be a bicategory and $\mathcal{D}$ a category. There is an equivalence of categories natural in $\mathcal{B}, \mathcal{D}$

$$\mathrm{Fun}(\mathrm{THH}(\mathcal{B}), \mathcal{D}) \simeq \mathcal{S}\mathrm{ha}(\mathcal{B}, \mathcal{D})$$

categories

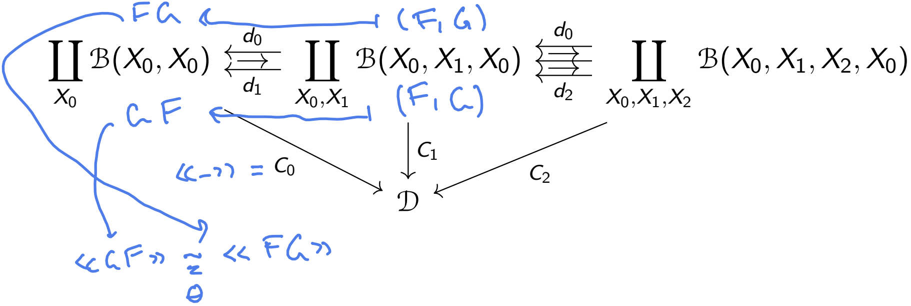
Obj <<->>, θ : B → D

<<->> : ∐ B(x,x) → D

The equivalence is very explicit given by functors going in both directions with one side strictly composing to the identity.

How to go from THH to shadows via cocones

A functor $\mathrm{THH}(\mathcal{B}) \to \mathcal{D}$ is the data of a pseudo-cocone



and we can already recognize the data of a shadow in this diagram!

Gist of Proof: Pseudocones

Let $\Delta_{\leq 2}$ be the truncated simplex category with objects $[0], [1], [2]$. $\mathrm{THH}(\mathcal{B})$ is defined via pseudo-colimit and so a functor out of it corresponds to solving the following pseudo-lifting problem:

$$\begin{array}{ccc} (\Delta_{\leq 2})^{op} & \xrightarrow{\coprod_{X_0,\dots,X_n} \mathcal{B}(X_0, \dots, X_n, X_0)} & \mathcal{C}\mathrm{at} \\ \downarrow & \nearrow & \\ ((\Delta_{\leq 2})^{op})^{\triangleright} & & \end{array}$$

The gist of the proof is recognizing that the conditions of a shadow are precisely the obstructions to such pseudo-lift, which uses a lot of ideas by Street and Lack.

Implications

Let us look at several implications

1. Why shadows and THH are related
2. Morita Invariance
3. $(\infty, 1)$-Categorical Shadow

Shadows are just right!

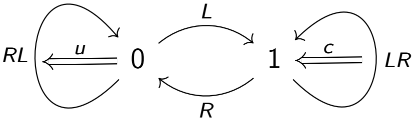
$$\mathrm{Fun}(\mathrm{THH}(\mathcal{B}),-) \simeq \mathrm{Sha}(\mathcal{B},-)$$

We have seen before (and there are many other examples) that shadows are very effective in studying THH, Morita invariance and other phenomena.

Our proof gives a conceptual reason for this relation by characterizing shadows via THH again.

The Free Adjunction Bicategory

Let $\mathcal{A}\mathrm{dj}$ be the free adjunction 2-category, meaning it is the free 2-category on the diagram



and relations

$$cL \circ Lu = \mathrm{id} \quad uR \circ Rc = \mathrm{id}.$$

It has the property that a functor $F : \mathcal{A}\mathrm{dj} \to \mathcal{B}$ is precisely a Morita dual diagram in $\mathcal{B}$.

From free Adjunctions to Morita Invariance

Now for a bicategory $\mathcal{B}$ we get a functor

$$\mathrm{THH} : \mathrm{Fun}(\mathcal{A}\mathrm{dj}, \mathcal{B}) \to \mathrm{Fun}(\mathrm{THH}(\mathcal{A}\mathrm{dj}), \mathrm{THH}(\mathcal{B})).$$

The objects on the left hand side are precisely the Morita dual pairs, so it would suffice to relate the right hand side to morphisms in $\mathrm{THH}(\mathcal{B})$, which would require understanding $\mathrm{THH}(\mathcal{A}\mathrm{dj})$.

THH of the Free Adjunction Category

The computation of $\mathrm{THH}(\mathcal{A}\mathrm{dj})$ is quite challenging, so we compute an approximation thereof. Let $\mathcal{A}\mathrm{dj}_{\leq 0}$ be the 2-category with the same generating 0, 1, 2-morphisms and with the additional relations $LRL = L$, $RLR = R$.

Proposition (Hess-R.)

There is an equivalence of categories

$$\mathrm{THH}(\mathcal{A}\mathrm{dj}_{\leq 0}) \simeq [2] = \{0 \leq 1 \leq 2\}$$

Moreover, the evident projection functor $\mathrm{THH}(\mathcal{A}\mathrm{dj}) \to \mathrm{THH}(\mathcal{A}\mathrm{dj}_{\leq 0}) \simeq [2]$ *has a section.*

Morita Invariance of Shadows via THH

$s_1 [2] \to \mathrm{THH}(\mathcal{A}\mathrm{dj})$

Precomposing with this section gives us the desired map

$$\mathrm{Fun}(\mathcal{A}\mathrm{dj}, \mathcal{B}) \xrightarrow{\mathrm{THH}} \mathrm{Fun}(\mathrm{THH}(\mathcal{A}\mathrm{dj}), \mathrm{THH}(\mathcal{B})) \xrightarrow{s}$$

$$\mathrm{Fun}([2], \mathrm{THH}(\mathcal{B})) \to \mathrm{Fun}([1], \mathrm{THH}(\mathcal{B}))$$

that takes the Morita dual (C, D, F, G, c, u) in $\mathcal{B}$ to the morphism

$$\mathrm{id}_C \xrightarrow{u} GF \cong FG \xrightarrow{c} \mathrm{id}_D$$

and functoriality implies that if the Morita dual is a Morita equivalence, then this morphism is an isomorphism.

Cool fact: This proof holds in a lot of places!

$(\infty, 1)$-Categorical Shadows

$\mathrm{Fun}(\mathrm{THH}(\mathcal{B}), \mathcal{D}) \simeq \mathrm{Sha}(\mathcal{B}, \mathcal{D})$

$\amalg \mathcal{B}(x,x) \to \mathcal{D}$

Using the main result we can also present the following definition:

Definition

Let $\mathcal{B}$ be an $(\infty, 2)$-category and $\mathcal{D}$ an $(\infty, 1)$-category. Then an $(\infty, 1)$-shadow is a functor of $(\infty, 1)$-categories $\mathrm{THH}(\mathcal{B}) \to \mathcal{D}$.

These notions are all well-defined due to work of Berman and in fact hold even if enriched over a symmetric monoidal ∞-category.

The next natural step is then to construct examples thereof.

The Future

1. Lift THH to an $(\infty, 1)$-shadow.
2. Tricategorical Shadows.
3. Construct coTHH as an $(\infty, 1)$-shadow.

Let's explain the first two in more detail.

Traces of Spectrally Enriched ∞-Categories

We saw before that one key example of a shadow is THH itself. Using the outline we now can formulate the appropriate homotopy coherent version

Question

Let $\mathrm{Mod}_{\mathrm{Sp}}$ be the $(\infty, 2)$-category of spectrally enriched bimodules. Prove there exists a functor of $(\infty, 1)$-categories

$$\mathrm{THH} : \mathrm{THH}(\mathrm{Mod}_{\mathrm{Sp}}) \to \mathrm{Sp}$$

such that at level of homotopy categories

$$\mathrm{Ho}(\mathrm{THH}) : \mathrm{HoTHH}(\mathrm{Mod}_{\mathrm{Sp}}) \to \mathrm{HoSp}$$

it recovers the shadow constructed by Ponto and Campbell.

Tricategorical Shadows

Question

Can we use similar steps to construct a tricategorical generalization of a shadow as the minimal data required to construct a lift

$$
\begin{array}{ccc}
(\Delta_{\leq 3})^{op} & \xrightarrow{\displaystyle\coprod_{X_0,\dots,X_n} \mathcal{B}(X_0, \dots, X_n, X_0)} & \mathcal{B}i\mathcal{C}at \\
\big\downarrow & \nearrow & \\
((\Delta_{\leq 3})^{op})^{\triangleright} & &
\end{array}
$$

Thank you! The End!

Thank you for your time!

For more details ...

- ... see the paper: *Shadows are Bicategorical Traces*, arXiv:2109.02144
- ... ask me: *nima.rasekh@epfl.ch*