Non-standard Truncations

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Algebraic Topology is great, but ...

Here are the axioms of this talk:

- Homotopy theory is fascinating!
- Spaces and Algebraic Topology are great!
- Homotopy groups are worth computing!

... it relies on set theory!

Algebraic topology is developed in the context of **sets**, meaning:

1. A topological space is a **set** with a topology.
2. A Kan complex is a simplicial **set** that satisfies a lifting condition.

More precisely: The sub-category of **discrete objects** is equivalent to sets. This causes issues!

Set Based Natural numbers!

We define *spheres* using an inductive process:

1. $S^{-1} = \emptyset$
2. $S^n = \Sigma S^{n-1}$.

Using this argument we can define the spheres S^n for $n \in \{0, 1, 2, ...\}$. Then using spheres we can define **homotopy groups** and **truncation levels**.
Our choice of natural number is restricted to the *set of natural numbers* $\mathbb{N} = \{0, 1, 2, ...\}$.

Alternative foundations

- Want to develop algebraic topology in a setting other than the category of sets.
- We focus on one example: **filter products**.
- Needs **filters**.

Filters

Let I be a set. A a filter should be thought of a collection of *"large"* subsets of I.

Definition

A subset $\mathfrak{F} \subseteq P(I)$ is called a **filter** if

1. $I \in \mathfrak{F}$ (The total subset is large)
2. $U, V \in \mathfrak{F} \Rightarrow U \cap V \in \mathfrak{F}$ (finite intersection of large subsets is large)
3. $U \in \mathfrak{F}, U \subseteq V \Rightarrow V \in \mathfrak{F}$ (supersets of large subsets are large)

Filter Products

Let $\mathcal{F}$ be a filter on I. Define the category $\prod_{\mathcal{F}} \mathcal{S}\mathsf{et}$ as follows:

- **Objects:** Indexed sets $(S_i)_{i \in I}$.
- **Morphisms:** Partially indexed morphisms up to equality on large subsets.

$$\mathrm{Hom}_{\prod_{\mathcal{F}} \mathcal{S}\mathsf{et}}((S_i)_{i \in I}, (T_i)_{i \in I}) = \coprod_{J \in \mathcal{F}} \left(\prod_{i \in J} \mathrm{Hom}(S_i, T_i) \right) / \sim$$

$$(f_i) \sim (g_i) \Leftrightarrow \exists J \in \mathcal{F}(\forall i \in J(f_i = g_i))$$

Filter Products Finitely Reasonable

The category $\prod_{\mathcal{F}} \mathcal{S}\mathrm{et}$ satisfies:

- It has **finite limits** and **colimits**: They are computed level-wise.
- It is **locally Cartesian close**d (internal mapping objects): Also computed level-wise.
- It has a **natural number object**: Given level-wise.

Ultraproducts

If $\mathcal{U}$ is a maximal non-trivial filter (also called **ultrafilter**) then the category $\prod_{\mathcal{U}} \mathcal{S}et$ satisfies:

1. It is **Boolean**: The final object 1 has no non-trivial subobjects.
2. It is **generated by the final object**: Two morphisms $f, g : X \to Y$ are equal if and only if $f(x) = g(x)$ for all $x : 1 \to X$.
3. If $\mathcal{U}$ is also **non-principal**, then $\prod_{\mathcal{U}} \mathcal{S}et$ does not have infinite colimits.

So $\prod_{\mathcal{U}} \mathcal{S}et$ has relevant features of the category of sets, while not being equivalent to it, meaning we get an alternative foundation!

Examples of Filter products!

Let's focus on an example:

Example

- $I = \mathbb{N}$
- $\mathcal{F} =$ cofinite subsets (*Frechet filter*)

$\mathcal{F}$ is not an *ultrafilter* (it doesn't include even or odd numbers), but is included in some ultrafilter.

We want to show that $\prod_{\mathcal{F}} \mathcal{S}\mathrm{et}$ has new natural numbers! That requires us understanding *natural number objects* in filter products.

Examples of Filter products! More details!

Let's be more detailed: $\prod_{\mathcal{F}} \mathcal{S}\text{et}$ has

- **Objects:** $(S_n)_{n\in\mathbb{N}}$
- **Morphisms:** $f : (S_n)_{n\in\mathbb{N}} \to (T_n)_{n\in\mathbb{N}}$ is defined on a cofinite subset $\Leftrightarrow \exists N \in \mathbb{N}(f_n : S_n \to T_n)_{n>N}$.

$$(f_n)_{n>N_1} \sim (g_n)_{n>N_2} \Leftrightarrow \exists N_3 \forall n > N_3(f_n = g_n)$$

We want to show that $\prod_{\mathcal{F}} \mathcal{S}\text{et}$ has weird natural numbers! That requires us understanding *natural number objects* in filter products.

Natural Numbers in Filter Quotients

Recall the *natural number object* in $\prod_{\mathbb{N}} \mathcal{S}\mathrm{et}$ corresponds to the sequence of sets $(\mathbb{N}, \mathbb{N}, ...)$.

Definition

A **natural number** is an element in $\mathrm{Hom}_{\prod_{\mathcal{F}} \mathcal{S}\mathrm{et}}((1), (\mathbb{N}))$, meaning it is an equivalence class $[(\{a_1\}, \{a_2\}, \{a_3\}, ...)]$, where two natural numbers are equal if they are "eventually equal".

Standard Natural Numbers

Some natural numbers look very familiar:

1. $[(\{0\}, \{0\}, \{0\}, \{0\}, \{0\}, ...)]$
2. $[(\{3\}, \{4\}, \{0\}, \{0\}, \{0\}, ...)]$
3. $[(\{1\}, \{2\}, \{1\}, \{2\}, \{1\}, ...)] =$

Standard Natural Numbers

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1. $[(\{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \ldots)]$
2. $[(\{3\}, \{4\}, \{0\}, \{0\}, \{0\}, \ldots)]$
3. $[(\{1\}, \{2\}, \{1\}, \{2\}, \{1\}, \ldots)] =$
   $[(\{1\}, \emptyset, \{1\}, \emptyset, \{1\}, \ldots)] \coprod [(\emptyset, \{2\}, \emptyset, \{2\}, \emptyset, \ldots)]$

$$\begin{array}{ccc}
(1, \emptyset, 1, \ldots) & \hookrightarrow & (1, 1, 1, \ldots) \\
\downarrow & \ulcorner & \downarrow {\scriptstyle (\{1\},\{2\},\{1\},\ldots)} \\
(1, 1, 1, \ldots) & \xrightarrow[(\{1\},\{1\},\{1\},\ldots)]{} & (\mathbb{N}, \mathbb{N}, \mathbb{N}, \ldots)
\end{array}$$

$$\begin{array}{ccc}
(\emptyset, 1, \emptyset, \ldots) & \hookrightarrow & (1, 1, 1, \ldots) \\
\downarrow & \ulcorner & \downarrow {\scriptstyle (\{1\},\{2\},\{1\},\ldots)} \\
(1, 1, 1, \ldots) & \xrightarrow[(\{2\},\{2\},\{2\},\ldots)]{} & (\mathbb{N}, \mathbb{N}, \mathbb{N}, \ldots)
\end{array}$$

Some natural numbers are standard, ...

These natural numbers are known as **standard**, meaning we can cover them by **successor numbers** $[(\{n\},\{n\},\{n\},\{n\},\{n\},\ldots)]$.

Lemma

If the countable coproduct of the final object exists then all natural numbers are standard, meaning the maps

$$\{(\{n\},\{n\},\{n\},\ldots):(1,1,1,\ldots)\to(\mathbb{N},\mathbb{N},\mathbb{N},\ldots)\}_{n\in\mathbb{N}}$$

are jointly surjective.

For example, this holds in $\mathcal{S}\mathrm{et}^{\mathbb{N}}$. On the other side, we want to see examples of non-standard natural numbers in $\prod_{\mathcal{F}}\mathcal{S}\mathrm{et}$.

... but some are non-standard!

The natural number $(\{1\},\{2\},\{3\},...)$ is not standard. Indeed we have

$$(\{1\},\{2\},\{3\},...) \cap (\{n\},\{n\},\{n\},...) =$$
$$(\emptyset,\emptyset,...,\emptyset,\{n\},\emptyset,...) = (\emptyset,\emptyset,...)$$

Hence, $(\{1\},\{2\},\{3\},...)$ has no non-trivial intersection with any successor natural number.

... but some are non-standard! Categorical Version!

Phrased categorically, for all *n* we have pullback squares

$$\begin{array}{ccc} \emptyset & \longrightarrow & (1,1,\ldots) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (\{1\},\{2\},\ldots)} \\ (1,1,\ldots) & \underset{(\{n\},\{n\},\ldots)}{\longrightarrow} & (\mathbb{N},\mathbb{N},\ldots) \end{array}$$

implying that $(\{1\},\{2\},\{3\},\ldots)$ is disjoint from the standard natural numbers.

This cannot happen in $\mathcal{S}\mathrm{et}^{\mathbb{N}}$, for example.

Some Theory: Elementary Toposes

There is a whole study of categories giving us alternative foundations to mathematics: **elementary toposes**. They have been studied extensively by many smart people with many interesting results.

But we are topology people, so why would we care about any of this?

Moving up to topology

We can generalize the definition from set theory to topology. Let $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$ be the category given as:

1. **Objects:** indexed Kan complexes $(X_n)_{n\in\mathbb{N}}$.
2. **Morphisms:** A morphism $f : (X_n)_{n\in\mathbb{N}} \to (Y_n)_{n\in\mathbb{N}}$ are morphisms $(f_n)_{n>N}$ that are eventually equal.
3. **Equivalences:** In particular a morphism f is an equivalence if and only if it is eventually an equivalence.

What can we say about its homotopy theory?

Theoretical Interlude I

We want to think of $\mathcal{T}\mathrm{op}$ as a **Kan enriched category**, which is a model for $(\infty, 1)$-categories. We will not go into too many details. Here is just a theoretical summary:

1. Given an $(\infty, 1)$-category (*quasi-category, complete Segal space, Kan enriched category*) $\mathcal{C}$ and a filter $\mathcal{F}$ on a set I, we can construct the filter product $\prod_{\mathcal{F}} \mathcal{C}$ as a direct construction or filtered colimit.
2. If $\mathcal{C}$ has finite (co)limits then so does $\prod_{\mathcal{F}} \mathcal{C}$ and it is defined level-wise.
3. If $\mathcal{C}$ has a natural number object, then $\prod_{\mathcal{F}} \mathcal{C}$ also has one defined level-wise.

Theoretical Interlude II

Definition

Given a finitely complete $(\infty, 1)$-category $\mathcal{C}$ an object X is 0-**truncated** (**discrete**) if the diagonal $\Delta : X \to X \times X$ is mono. The subcategory of 0-truncated objects is denoted $\tau_0\mathcal{C}$.

We have $\tau_0\mathcal{T}\mathrm{op} \simeq \mathcal{S}\mathrm{et}$.

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We have $\tau_0\mathcal{T}\mathrm{op} \simeq \mathcal{S}\mathrm{et}$.

Lemma

We have an equivalence $\tau_0(\prod_{\mathcal{F}} \mathcal{C}) \simeq \prod_{\mathcal{F}} \tau_0\mathcal{C}$. *In particular* $\tau_0(\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}) \simeq \prod_{\mathcal{F}} \mathcal{S}\mathit{et}$.

Hence $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$ has a different underlying logic as $\mathcal{T}\mathrm{op}$, despite other similarities.

Spheres and Truncated Objects

Hence $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$ has the same natural numbers as $\prod_{\mathcal{F}} \mathcal{S}\mathrm{et}$. Fix a natural number $\{a_n\} = [(\{a_1\}, \{a_2\}, ...)]$.

- Define the sphere $S^{(\{a_n\})} = (S^{a_1}, S^{a_2}, ...)$.
- For a given space $X = (X_1, X_2, ...)$, we say X is (a_n)-**truncated** if the induced map

$$X \to X^{S^{(a_n+1)}}$$

is an equivalence in $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$.

Internal vs. External Truncations: Some agree...

Here is a basic result about 0-truncated objects.

Lemma

An object X in $\mathcal{C}$ in 0-truncated if and only if for all objects Y, the mapping space $\mathrm{Map}(Y, X)$ is 0-truncated (equivalent to a set).

Internal vs. External Truncations: Some agree...

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This generalizes to *n*-truncations and applied to $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$ gives us

Lemma

Assume $(a_1, a_2, \ldots)$ is eventually equal to L. Then an object X in $\prod_{\mathcal{F}} \mathcal{T}\mathrm{op}$ is (a_n)-truncated if and only if the mapping space $\mathrm{Map}(-, X)$ is L-truncated.

... and some not!

On the other hand the natural number $(1, 2, 3, ...)$ is non-standard. So we cannot express

$$\text{“}X \textit{ is } (1, 2, 3, ...)\textit{-truncated”}$$

via $\mathrm{Map}(-, X)$ in spaces.

Similarly, we can use natural numbers to define homotopy groups, so we have additional homotopy groups in this category.

This cannot happen if the category has infinite coproducts and warrants further study!

Upshot

1. Topology is great and has been able to prove many interesting results.
2. However, the current work has restricted itself to a certain kind of foundations.
3. This leaves us with the question what else is out there? The filter product is one example, but clearly we expect more.
4. This general effort is known as **synthetic algebraic topology** in **homotopy type theory** or **elementary higher topos theory**.
5. There is much work left! We are barely in the 50s!

Where do we go from here?

Focusing on filter products again:

1. Can we compute these new homotopy groups?
2. Can we use them to say anything meaningful about the standard homotopy groups?
3. How about further homotopy theory? For example what is an appropriate notion of spectra?
4. What implication does this have for operads and algebraic structure?

Our first step is to show that filter product $(\infty, 1)$-categories have induced model structures.

The End!

For more details see:

- **Filter Quotients and Non-Presentable $(\infty, 1)$-Toposes**, arXiv:2001.10088
- **An Elementary Approach to Truncations**, arXiv:1812.10527

Thank you!

Questions?