ICERM SCHOOL NOTES: COMPLETE SEGAL SPACES

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ABSTRACT. These are lecture notes for the ICERM 2026 Summer School "Teaching Higher Category Theory with Computers".

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0. INTRODUCTION

This summer school aims to teach formalization of higher category theory via the proof assistant `Rzk`. This foundation relies on the type theory sHoTT, which itself has semantics in complete Segal spaces. These notes introduce complete Segal spaces to give participants some context.

0.1. **Motivation.** Category theory is a mathematical framework that allows us to study mathematical objects and their relationships in a very general way. By definition it contains a lot of *uniqueness statements*:

(1) Composition of two morphisms is unique.
(2) Composition of three morphisms is unique (associativity).

There are also many slightly weaker *unique up to isomorphism statements*:

(1) A terminal (initial) object is unique up to isomorphism.
(2) A product (coproduct) is unique up to isomorphism.

Unfortunately, this setup does not work in all settings. Whenever we are in a homotopic setting:

(1) Algebraic topology
(2) Derived geometry
(3) ...

All of these uniqueness statements have to be adjusted. The collective response to this issues is known as *higher category theory*, and here we focus on ∞-*category theory*. This theory has been approached using different methods:

Date: August 2026.

(1) **Analytic:** Traditionally, the first approach consists of trying to explicitly incorporate relevant structure.
(2) **Synthetic:** More recently, we see more examples of a second approach, that instead changes the underlying foundation.

Each approach has its benefits and drawbacks:

Approach	**Benefits**	**Drawbacks**
Analytic	• No need to learn a new foundation • Directly connects to other areas of mathematics	• Requires more technical machinery • Proof methods diverge from intuition
Synthetic	• Structures more immediate and amenable to formalization • Proofs match intuition	• Requires significant theoretical development • More difficult relating to other areas of mathematics

This lecture series will focus on the first approach, the second parallel lecture series will pursue the second approach. The hope is that we can combine the best of both worlds:

- The synthetic approach allows you to actually state and prove non-trivial results.
- The analytic approach allows to connect the new foundation to a more classical understanding of mathematics, that is more familiar.

0.2. **How to read the notes.** This note is a more extensive version of the material that we cover in the lectures.

- One can pick up the notes and start reading them in a linear fashion. However, one can also pick definitions or concepts that are less familiar or of particular interest.
- The exercises are primarily meant to further your understanding of the material.

The first lecture focuses on the basics of the theory of simplicial sets.
homotopy theory and category of simplicial sets. In these notes this corresponds to:

- Section 1: Foundations of simplicial sets, important examples and notations

The second lecture focuses on the interaction of simplicial sets with homotopy theory and category theory.

- Section 2: Important applications of simplicial sets: homotopy theory (via singular functor) and category theory (via the nerve construction)

The third lecture focuses on the homotopy theory of simplicial sets and introduces simplicial spaces.

- Section 4: Kan fibrations and their role in simplicial homotopy theory
- Section 6: Contractible simplicial sets and trivial Kan fibrations
- Section 3: Foundations of simplicial spaces, important examples and notations

The fourth lecture focuses Reedy fibrancy and introduces Segal spaces

- Section 5: The theory of Reedy fibrations
- Section 7: The definition of Segal spaces

The fifth focuses on the category theory of Segal spaces and the completeness condition.

- Section 8: Various characterizations of equivalences in Segal spaces
- Section 9: The completeness condition for Segal spaces

Further material that is not covered in the lectures but is included in the notes and might be of interest to the reader:

- Section 10: Review of Yoneda lemma for classical categories and discrete fibrations
- Section 11: Motivating and introducing left fibrations

0.3. **Prerequisites.** The notes require a basic understanding of category theory, corresponding to [ML98, Sections 1–3] or [Rie16, Chapters 1–3]. More explicitly, here is a (non-exhaustive) list of concepts that are used in the notes:

- Categories, functors, natural transformations, and equivalences of categories.
- Examples of categories: sets, topological spaces. Also, posets as categories.
- Products of categories and functor categories.
- Fully faithful, essentially surjective functors, and equivalences of categories.
- The Yoneda lemma and the Yoneda embedding, the restricted Yoneda embedding along fully faithful restrictions.
- Limits and colimits, in particular pullbacks and pushouts.
- Limits and colimits in functor categories, in particular the fact that they are computed levelwise.

0.4. **Notation.** The categorical background follows conventional notation, as in [Rie16]. In particular, we have the following notational conventions:

- We denote the category of sets by $\mathcal{S}\mathrm{et}$.
- We denote the category of categories by $\mathcal{C}\mathrm{at}$.
- We denote the category of topological spaces by $\mathcal{T}\mathrm{op}$.
- We use the notation $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the functor category from $\mathcal{C}$ to $\mathcal{D}$, with objects functors and morphisms natural transformations.
- We denote by $[n]$ the category given by the poset $\{0 \leq 1 \leq \cdots \leq n\}$.
- We denote by $I(1)$ the groupoid with 2 objects and a unique morphism between any two objects.

Warning 0.1. There is one major deviation from standard notation, namely regarding the notation of representable simplicial sets (and spaces).

- In the category of simplicial sets $s\mathcal{S}\mathrm{et}$ (Definition 1.8), we denote the standard n-simplex by $\mathbb{D}^n$ (Definition 1.16), its boundary by $\partial\mathbb{D}^n$ (Definition 1.33), the k-th horn by $\mathbb{L}^n_k$ (Definition 1.37), and the spine by Sp^n (Definition 1.40)[1].
- In the category of simplicial spaces $s\mathcal{S}$ (Definition 3.2), we denote the two standard simplices by Δ^n and $\mathbb{D}^l$. For Δ^n its boundary is denoted by $\partial\Delta^n$, the k-th horn by Λ^n_k, and the spine by Sp^n, whereas the sub-objects of $\mathbb{D}^l$ are denoted as above (Notation 3.10)[2].

This unusual notation is chosen to better match our notation with the syntax of `Rzk`.

1. Simplicial Sets

The goal of this section is to define the magical simplex category $\mathbb{\Delta}$ and the category of simplicial sets, $s\mathcal{S}\mathrm{et}$. In the next section we will then see how simplicial sets relate to categories and topological spaces. For more extensive introductions, assuming more category theory, see [Rie11, Fri12] or [GJ09, Section I.1].

1.1. **First Definitions.** Let us start with the definition of the simplex category $\mathbb{\Delta}$ and simplicial sets.

Definition 1.1 (Simplex Category)**.** Let $\mathbb{\Delta}$ be the category defined as follows:

- $\mathrm{Obj}_{\mathbb{\Delta}} := \{[n] \mid n \geq 0\}$, where $[n] := \{0 \leq \cdots \leq n\}$ is the linearly ordered set of integers from 0 to n.
- For $[m], [n]$ in $\mathrm{Obj}_{\mathbb{\Delta}}$, $\mathrm{Hom}_{\mathbb{\Delta}}([m], [n])$ is the set of all functors (i.e., non-decreasing maps) from $[m]$ to $[n]$.
- The composition and identity are the regular compositions and identities of functions. Equivalently, these are the composition of functors and identity functors in $\mathcal{C}\mathrm{at}$.

[1] These are usually denoted as Δ^n, $\partial\Delta^n$, Λ^n_k, and Sp^n respectively

[2] Here, there is no one standard notation throughout the literature. The original source regarding complete Segal spaces [Rez01] introduced $F(n)$ for Δ^n and $\Delta[l]$ for $\mathbb{D}^l$. Other sources take a combined approach and denote $\Delta^n \times \mathbb{D}^l$ by $\Delta[n, l]$ or $\Delta^n \square \Delta^l$.

Notation 1.2. By definition a morphism $[m] \to [n]$ in $\mathbb{\Delta}$ is precisely a non-decreasing chain $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m \leq n$, where a_i denotes the image of $i \in [m]$. We will hence denote such a morphism by $a_0 a_1 ... a_m$.

In general $\mathbb{\Delta}$ has too many morphisms. Indeed, using a basic counting argument from combinatorics we see that $\mathrm{Hom}_{\mathbb{\Delta}}([m],[n])$ has $\binom{m+n+1}{n}$ elements. However, we can reduce arbitrary morphisms to a smaller set of "generating morphisms" via composition.

Notation 1.3. Some morphisms in $\mathbb{\Delta}$ merit their own notation.

- For each $n \geq 0$ and $0 \leq i \leq n+1$ there is a unique injective map
$$d^i \colon [n] \to [n+1]$$
whose image does not contain i.
- For each $n \geq 1$ and $0 \leq i \leq n-1$ there is a unique surjective map
$$s^i \colon [n] \to [n-1]$$
whose pre-image over i is $\{i, i+1\}$ and other pre-images have one element.

Why do we specifically single out these morphisms? It turns out these morphisms are enough to get everything else back. The proof is a somewhat tedious combinatorial argument, and hence we refer the reader to [ML98, Section VII.5].

Lemma 1.4. *An arbitrary morphism $\delta \colon [m] \to [n]$ in $\mathbb{\Delta}$ can be uniquely written as a composition of the form*
$$\delta = d^{i_1} d^{i_2} \cdots d^{i_k} s^{j_1} \cdots s^{j_l},$$
where $0 \leq i_k < \cdots < i_2 < i_1 \leq n$ and $0 \leq j_1 < j_2 < \cdots < j_l < m$.

This means we only need to better understand how the morphisms d^i and s^i interact with each other. Here we can use the explicit description given in Notation 1.3, to obtain the *cosimplicial identities*.

Lemma 1.5 (Cosimplicial identities)**.** *The morphisms d^i and s^i satisfy the following identities:*

(1) $d^j d^i = d^i d^{j-1}$ *for* $i < j$.
(2) $s^j s^i = s^i s^{j+1}$ *for* $i \leq j$.
(3) $s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ \mathrm{id} & i = j, j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$

Remark 1.6. Combining Lemmas 1.4 and 1.5 the category $\mathbb{\Delta}$ can be depicted as follows:

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & {[1]} \arrow[l, "s^0" description] \arrow[r, shift left=4, "d^0"] \arrow[r] \arrow[r, shift right=4, "d^2"'] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \end{tikzcd} .$$

We can now use the category $\mathbb{\Delta}$ to define simplicial sets. Here we use the category of sets, and functor categories.

Definition 1.7 (Simplicial Object)**.** Let $\mathcal{C}$ be a category. A *simplicial object* in $\mathcal{C}$ is a functor $X \colon \mathbb{\Delta}^{op} \to \mathcal{C}$.

Definition 1.8 (Simplicial Set)**.** A *simplicial set* is a simplicial object in $\mathcal{S}$et. The category of simplicial sets, denoted $s\mathcal{S}$et, is the functor category $\mathrm{Fun}(\mathbb{\Delta}^{op}, \mathcal{S}\mathrm{et})$.

Notation 1.9. Given a category $\mathcal{C}$ and a simplicial object $X \colon \mathbb{\Delta}^{op} \to \mathcal{C}$, for any map $\delta \colon [m] \to [n]$ in $\mathbb{\Delta}$, we denote the corresponding map $X(\delta) \colon X_n \to X_m$ in $\mathcal{C}$ by δ^*. In particular, for the morphisms d^i and s^i we denote the corresponding maps by $d_i = (d^i)^*$ and $s_i = (s^i)^*$.

Following Lemmas 1.4 and 1.5, we can now give a more explicit description of what a simplicial set is.

Lemma 1.10. *Let $\mathcal{C}$ be a category. A simplicial object X is uniquely determined by the following data:*

- *A choice of objects* $X_0 = X([0]), X_1 = X([1]), \ldots.$
- *For each* $n \geq 0$ *and* $0 \leq i \leq n+1$ *morphisms* $X(d^i) = d_i \colon X_{n+1} \to X_n$.
- *For each* $n \geq 1$ *and* $0 \leq i \leq n-1$ *there are morphisms* $X(s^i) = s_i \colon X_{n-1} \to X_n$.
- *Satisfying the* simplicial identities*:*
  - $d_i d_j = d_{j-1} d_i$ *for* $i < j$.
  - $s_i s_j = s_{j+1} s_i$ *for* $i \leq j$.
  - $d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$

Remark 1.11. Following this lemma, similar to Remark 1.6, we can depict a simplicial set X as follows:

$$\begin{tikzcd} X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left] \arrow[r, shift right] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3, "d_2"] \arrow[r, shift left] \arrow[r, shift right] \arrow[r] & \cdots \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \end{tikzcd}$$

Notice all arrows are reversed as this functor is mapping out of Δ^{op}.

Definition 1.12 (Degenerate Simplex)**.** Let X be a simplicial set. An n-simplex σ in X_n is called *degenerate* if there exists a simplex τ in X_{n-1} and $0 \leq i \leq n-1$ such that $\sigma = s_i(\tau)$. Otherwise, σ is called *non-degenerate*.

Intuition 1.13. Geometrically, we imagine a simplicial set X as a recipe for constructing a topological space with a notion of "direction". Here X_0 is the set of vertices, non-degenerate elements in X_1 are the set of edges (which is where the direction comes from), non-degenerate elements in X_2 are the set of triangles, and so on. The maps $d_0, d_1 \colon X_1 \to X_0$ specify the source and target of each edge, while the maps $d_0, d_1, d_2 \colon X_2 \to X_1$ specify the edges that form the boundary of each triangle. The degenerate simplices do not manifest geometrically, but do play an important role (see Remark 1.46).

Let us see one simple example of a simplicial set.

Definition 1.14 (Constant simplicial set)**.** A simplicial set X is *constant* if all s_i, d_i are bijections.

Every set S gives rise to a constant simplicial set, as we see in the next example.

Example 1.15. Let S be a set. Define the constant simplicial set $S \colon \Delta^{op} \to \mathcal{S}et$ by $S_n = S$ for all $n \geq 0$, and all face and degeneracy maps are identity maps.

1.2. **Standard Simplices and the Yoneda Lemma.** Let us now look at representable functors in the context of simplicial sets.

Definition 1.16 (Standard simplex)**.** For a given $n \geq 0$, the *standard n-simplex* is the representable functor $\mathrm{Hom}_{\Delta}(-, [n])$, denoted by $\mathbb{D}^n$.

Remark 1.17. In Notation 1.2 we explained how we can depict elements in $\mathrm{Hom}_{\Delta}([m],[n]) = \mathbb{D}^n_m$ as a non-decreasing chain $a_0 \ldots a_m$. From this perspective, the map $d_i \colon \mathbb{D}^n_{m+1} \to \mathbb{D}^n_m$ is given by deleting the i-th element of the chain, while the map $s_i \colon \mathbb{D}^n_{m-1} \to \mathbb{D}^n_m$ is given by repeating the i-th element of the chain.

Remark 1.18. As the degeneracy maps s_i add repeating elements (Remark 1.17), an m-simplex $[m] \to [n]$ in $\mathbb{D}^n_m$ is non-degenerate if and only if the corresponding sequence is injective, meaning the chain $a_0 \ldots a_m$ has no repeated elements.

Example 1.19 (0-Simplex)**.** The standard 0-simplex $\mathbb{D}^0$ is the simplicial set defined by

$$\mathbb{D}^0_n := \mathrm{Hom}_{\Delta}([n],[0]) = \{0 \ldots 0\},$$

where in the last step we used Notation 1.2. This means we can depict $\mathbb{D}^0$ as follows:

$$\begin{tikzcd} \{0\} \arrow[r, "s_0" description] & \{00\} \arrow[l, shift right=2, "d_0 \ \cong"'] \arrow[l, shift left=2, "d_1 \ \cong"] \arrow[r, shift left] \arrow[r, shift right] & \{000\} \arrow[l, shift right=3, "d_0 \ \cong"'] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3, "d_2 \ \cong"] \arrow[r, shift left] \arrow[r, shift right] \arrow[r] & \cdots \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \end{tikzcd}$$

We have the following evident result about $\mathbb{D}^0$.

Lemma 1.20. *$\mathbb{D}^0$ is the terminal simplicial set, i.e., for every simplicial set X, there is a unique map $X \to \mathbb{D}^0$.*

Proof. Exercise 1.4 □

Intuition 1.21. $\mathbb{D}^0$ has a single non-degenerate simplex, namely 0 in $\mathbb{D}^0_0$. Hence, geometrically, $\mathbb{D}^0$ is a single point.

$$\overset{0}{\bullet}$$

Example 1.22 (1-Simplex)**.** The standard 1-simplex $\mathbb{D}^1$ is the simplicial set defined by

$$\mathbb{D}^1_n := \mathrm{Hom}_{\mathbb{\Delta}}([n],[1]) = \{0...0, 0...01, 0...011, ..., 1...1\},$$

where we again used Notation 1.2. This means we can depict $\mathbb{D}^1$ as follows:

$$\begin{tikzcd} \{0,1\} \arrow[r, "s_0"] & \{00,01,11\} \arrow[l, shift left=2, "d_0"'] \arrow[l, shift right=2, "d_1"] \arrow[r, shift left] \arrow[r, shift right] & \{000,001,011,111\} \arrow[l, shift left=3, "d_0\ \cong"'] \arrow[l, "\vdots"] \arrow[l, shift right=3, "d_2\ \cong"] \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] \end{tikzcd} .$$

Intuition 1.23. $\mathbb{D}^1$ has three non-degenerate simplices, namely 0 and 1 in $\mathbb{D}^1_0$ and 01 in $\mathbb{D}^1_1$. Hence, geometrically, $\mathbb{D}^1$ is a single edge connecting two vertices.

$$\overset{0}{\bullet} \xrightarrow{01} \overset{1}{\bullet}$$

Example 1.24 (2-Simplex)**.** The standard 2-simplex $\mathbb{D}^2$ is the simplicial set defined by

$$\mathbb{D}^2_n := \mathrm{Hom}_{\mathbb{\Delta}}([n],[2]) = \{0...0, 0...01, 0...011, ..., 2...2\},$$

where we again used Notation 1.2. This means we can depict $\mathbb{D}^2$ as follows:

$$\begin{tikzcd} \{0,1,2\} \arrow[r, "s_0"] & \{00,01,11,12,22,02\} \arrow[l, shift left=2, "d_0"'] \arrow[l, shift right=2, "d_1"] \arrow[r, shift left] \arrow[r, shift right] & \{000,001,011,111,112,122,222,002,022,012\} \arrow[l, shift left=3, "d_0"'] \arrow[l, "\vdots"] \arrow[l, shift right=3, "d_2"] \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] \end{tikzcd} .$$

Intuition 1.25. $\mathbb{D}^2$ has seven non-degenerate simplices, namely 0, 1, and 2 in $\mathbb{D}^2_0$, and 01, 12, and 02 in $\mathbb{D}^2_1$, and 012 in $\mathbb{D}^2_2$. Hence, geometrically, $\mathbb{D}^2$ is a single triangle with three edges and three vertices.

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[dr, "12"] & \\ \overset{0}{\bullet} \arrow[ur, "01"] \arrow[rr, "02"'] \arrow[rr, phantom, bend left=20, "012"] & & \overset{2}{\bullet} \end{tikzcd}$$

Having learned about the standard simplices, we now continue our theoretical analysis of simplicial sets, relying on the Yoneda lemma.

Lemma 1.26 (Yoneda lemma for simplicial sets)**.** *For a given $n \geq 0$ and a simplicial set X, there is a natural isomorphism $\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^n, X) \cong X_n$.*

We can in particular use the Yoneda lemma to understand the morphisms between standard simplices.

Corollary 1.27. *Let $n, k \geq 0$. There is a natural isomorphism $\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^n, \mathbb{D}^k) \cong \mathrm{Hom}_{\mathbb{\Delta}}([n],[k])$.*

Let us end with the observation that the standard simplices allow us to define mapping simplicial sets. Here, besides the Yoneda lemma, we also use opposite categories and products.

Definition 1.28 (Mapping simplicial set in $s\mathcal{S}et$)**.** Let X, Y be two simplicial sets. The *mapping simplicial set* $\mathrm{Map}(X,Y)$ is defined as the simplicial set

$$\mathbb{\Delta}^{op} \xrightarrow{\mathcal{Y}on^{op}} s\mathcal{S}et^{op} \xrightarrow{(X\times -)^{op}} s\mathcal{S}et^{op} \xrightarrow{\mathrm{Hom}(-,Y)} \mathcal{S}et.$$

Remark 1.29. Very explicitly we can characterize the n-simplices of $\mathrm{Map}(X,Y)$ as $\mathrm{Hom}(X \times \mathbb{D}^n, Y)$, and the face and degeneracy maps are induced by the corresponding maps of $\mathbb{D}^n$. In particular $\mathrm{Map}(X,Y)_0 = \mathrm{Hom}(X,Y)$, so the vertices of $\mathrm{Map}(X,Y)$ are precisely the maps from X to Y.

1.3. **Sub-Simplicial Sets.** Later on, beyond the standard simplices, we also need various sub-objects of $\mathbb{D}^n$.

Definition 1.30 (Sub-simplicial set)**.** Let X be a simplicial set. A *sub-simplicial set* Y of X is a natural monomorphism $Y \to X$.

Remark 1.31. More explicitly, for a simplicial set X, a sub-simplicial set Y of X consists of subsets $Y_n \subseteq X_n$ for every $n \geq 0$ such that for every $\delta\colon [m] \to [n]$ in $\mathbb{\Delta}$, the function $X(\delta)\colon X_n \to X_m$ restricts to a function $Y(\delta)\colon Y_n \to Y_m$.

One way to create sub-simplicial sets is by picking a subset of the 0-simplices.

Example 1.32. Let X be a simplicial set and let $S \subseteq X_0$ be a subset of the 0-simplices. Then the *sub-simplicial set generated by* S is the sub-simplicial set $\bar{S}$ in X, with $\bar{S}_n \subseteq X_n$ for every $n \geq 0$, given by

$$\bar{S}_n = \{\sigma \in X_n \mid \forall 0 \leq i \leq n(i^*\sigma \in S)\}.$$

Note, in particular $\bar{S}_0 = S$. In other words, $\bar{S}$ is the largest sub-simplicial set of X whose 0-simplices are precisely S.

There are, however, several specific sub-simplicial sets of interest, that we now review.

Definition 1.33 (Boundary)**.** For every $n \geq 0$, the *boundary* of $\mathbb{D}^n$, denoted $\partial\mathbb{D}^n$, is the sub-simplicial set of $\mathbb{D}^n$ with $\partial\mathbb{D}^n_m := \{\delta\colon [m] \to [n] \mid \mathrm{Im}(\delta) \neq [n]\}$. This is indeed a sub-simplicial set as non-surjective maps are closed under precomposition.

Example 1.34 (0-Boundary)**.** Every single simplex of $\mathbb{D}^0_n$ is given by a surjective map $[n] \to [0]$. Hence, $\partial\mathbb{D}^0$ is the empty simplicial set, meaning $\partial\mathbb{D}^0 = \emptyset$.

Example 1.35 (1-Boundary)**.** Following Remark 1.18, $\mathbb{D}^1$ has three non-degenerate simplices. Among those, $0, 1$ in $\mathbb{D}^1_0$ are not surjective, while 01 in $\mathbb{D}^1_1$ is surjective (following our notational convention it corresponds to the identity $[1] \to [1]$). Hence, we can depict $\partial\mathbb{D}^1$ as the simplicial set:

$$\begin{tikzcd} \{0,1\} \arrow[r, "s_0" description] & \{00,11\} \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=1] \arrow[r, shift right=1] & \{000,111\} \arrow[l, shift right=4, "d_0"'] \arrow[l] \arrow[l, shift left=4, "d_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift left=3] & \cdots \arrow[l, shift right=4] \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[l, shift left=4] \end{tikzcd} .$$

and geometrically, we have the following:

$$\begin{tikzcd} \overset{0}{\bullet} & \overset{1}{\bullet} \end{tikzcd}$$

Example 1.36 (2-Boundary)**.** Following Remark 1.18, $\mathbb{D}^2$ has seven non-degenerate simplices. Among those, $0, 1, 2$ in $\mathbb{D}^2_0$ and $01, 12, 02$ in $\mathbb{D}^2_1$ are not surjective, while 012 in $\mathbb{D}^2_2$ is the identity map and hence surjective. Hence, we can depict $\partial\mathbb{D}^2$ as the simplicial set:

$$\begin{tikzcd} \{0,1,2\} \arrow[r, "s_0" description] & \{00,01,11,12,22,02\} \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=1] \arrow[r, shift right=1] & \{000,001,011,111,112,122,222,002,022\} \arrow[l, shift right=4, "d_0"'] \arrow[l] \arrow[l, shift left=4, "d_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift left=3] & \cdots \arrow[l, shift right=4] \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[l, shift left=4] \end{tikzcd} .$$

and geometrically, we have the following:

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[dr, "12"] & \\ \overset{0}{\bullet} \arrow[ur, "01"] \arrow[rr, "02"'] & & \overset{2}{\bullet} \end{tikzcd}$$

Definition 1.37 (Horn)**.** Let $n \geq 1$ and let $0 \leq i \leq n$. The *i-th horn* of $\mathbb{D}^n$, denoted $\mathbb{L}^n_i$, is the sub-simplicial set of $\mathbb{D}^n$ with $(\mathbb{L}^n_i)_m := \{\delta\colon [m] \to [n] \mid \mathrm{Im}(\delta) \cup \{i\} \neq [n]\}$.

From the definition, we see that $\mathbb{L}^n_i$ is always contained in $\partial\mathbb{D}^n$.

Example 1.38 (1-Horns)**.** In the case $n = 1$, we have two horns, $\mathbb{L}^1_0$ and $\mathbb{L}^1_1$. $(\mathbb{L}^1_0)_0 \subseteq \mathbb{D}^1_0 = \{0, 1\}$ needs to exclude maps whose complement of the image only contains 0. Hence, $(\mathbb{L}^1_0)_0 = \{0\}$. Similarly, $(\mathbb{L}^1_1)_0 = \{1\}$. Hence $\mathbb{L}^1_0 \cong \mathbb{L}^1_1 \cong \mathbb{D}^0$, meaning geometrically we have a single vertex.

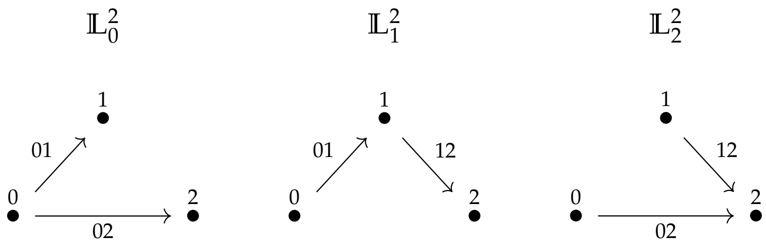
Example 1.39 (2-Horns)**.** In the case $n = 2$, we have three horns $\mathbb{L}^2_0$, $\mathbb{L}^2_1$, and $\mathbb{L}^2_2$. For $\mathbb{L}^2_0$, we make the following observations:

- $(\mathbb{L}^2_0)_0$ needs to exclude maps whose complement of the image only contains 0. But every map $[0] \to [2]$ has two elements in the complement of the image and hence $(\mathbb{L}^2_0)_0 = \{0, 1, 2\}$.
- $(\mathbb{L}^2_0)_1$ needs to exclude maps whose complement of the image only contains 0. That is precisely the map $12\colon [1] \to [2]$. Hence, $(\mathbb{L}^2_0)_1 = \{00, 01, 11, 02, 22\}$.

Hence, $\mathbb{L}^2_0$ is the simplicial set depicted as follows:

$$\{0,1,2\} \;\substack{\xleftarrow{\;d_0\;} \\ \xrightarrow{\;s_0\;} \\ \xleftarrow[\;d_1\;]{}}\; \{00,01,11,22,02\} \;\substack{\xleftarrow{\;d_0\;} \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \xleftarrow[\;d_2\;]{}}\; \{000,001,011,111,222,002,022\} \;\substack{\leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow}\; \cdots .$$

We can similarly describe $\mathbb{L}^2_1$ and $\mathbb{L}^2_2$. Geometrically, these three horns can hence be depicted as follows:



Definition 1.40 (Spine)**.** For every $n \geq 2$, the *spine* of $\mathbb{D}^n$, denoted $\mathbb{S}\mathrm{p}^n$, is the sub-simplicial set of $\mathbb{D}^n$ with $\mathbb{S}\mathrm{p}^n_m := \{\delta\colon [m] \to [n] \mid \delta(m) - \delta(0) \leq 1\}$. This is indeed a sub-simplicial set as such maps are closed under precomposition.

We can make the following two observations about the spine.

Lemma 1.41. *For every $n \geq 2$, $\mathbb{S}\mathrm{p}^n_0 = \mathbb{D}^n_0$.*

Proof. Exercise 1.8 □

Lemma 1.42. *For every $n, k \geq 2$, every element in $\mathbb{S}\mathrm{p}^n_k$ is degenerate.*

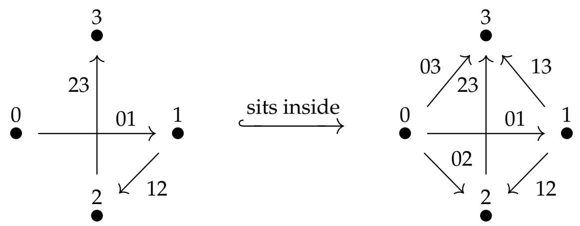
Proof. Exercise 1.9 □

Example 1.43 (2-Spine)**.** For $n = 2$, we can already evaluate $\mathbb{S}\mathrm{p}^2_0$ (Lemma 1.41) and $\mathbb{S}\mathrm{p}^2_2$ (Lemma 1.42), and hence focus on $\mathbb{S}\mathrm{p}^2_1$. We see that $\mathbb{S}\mathrm{p}^2_1 = \{00, 01, 11, 12, 22\}$ as the map $02\colon [1] \to [2]$ has $\delta(1) - \delta(0) = 2$ and hence does not satisfy the condition. Hence $\mathbb{S}\mathrm{p}^2 \cong \mathbb{L}^2_1$.

Example 1.44 (3-Spine)**.** For $n = 3$, we can already evaluate $\mathbb{S}\mathrm{p}^3_0$ (Lemma 1.41) and $\mathbb{S}\mathrm{p}^3_2$ (Lemma 1.42), and hence focus on $\mathbb{S}\mathrm{p}^3_1$. Similar to the previous example, we can hence see that

$$\mathbb{S}\mathrm{p}^3_1 = \{00, 01, 11, 12, 22, 23, 33\}.$$

Hence, geometrically, Sp^3 can be depicted as follows:



where the geometric shape of $\mathbb{S}\mathrm{p}^3$ was chosen in a manner to illustrate how it sits inside $\mathbb{D}^3$ as a sub-simplicial set.

1.4. **Limits and Colimits of Simplicial Sets.** Let us move on to the last topic, namely limits and colimits in $s\mathbb{S}\mathrm{et}$. Recall that limits and colimits in $s\mathbb{S}\mathrm{et}$ are computed level-wise. Here, we focus on explicit computations of limits and colimits of interest in $s\mathbb{S}\mathrm{et}$.

Example 1.45. The simplicial set $\mathbb{D}^1 \times \mathbb{D}^1$ is defined as the product of simplicial sets. Hence, for every $n \geq 0$, we have $(\mathbb{D}^1 \times \mathbb{D}^1)_n = \mathbb{D}^1_n \times \mathbb{D}^1_n$. This means that for the first three levels we have the following explicit computations:

- $(\mathbb{D}^1 \times \mathbb{D}^1)_0 = \{(0,0),(0,1),(1,0),(1,1)\}$
- $(\mathbb{D}^1 \times \mathbb{D}^1)_1 = \{(00,00),(00,01),(00,11),(01,00),(01,01),(01,11),(11,00),(11,01),(11,11)\}$
- $(\mathbb{D}^1 \times \mathbb{D}^1)_2 = \{(000,000),(000,001),(000,011),(000,111),(001,000),(001,001),(001,011),$ $(001,111),(011,000),(011,001),(011,011),(011,111),(111,000),(111,001),(111,011),(111,111)\}$

Interestingly, the non-degenerate simplices of $\mathbb{D}^1 \times \mathbb{D}^1$ are given by all 0-simplices, the five 1-simplices $(00,01)$, $(11,01)$, $(01,00)$, $(01,11)$, and $(01,01)$, and the two 2-simplices $(011,001)$ and $(001,011)$. Geometrically, we can hence depict $\mathbb{D}^1 \times \mathbb{D}^1$ as follows:

$$\begin{tikzcd} \overset{(0,0)}{\bullet} \arrow[r, "{(00,01)}"] \arrow[d, "{(01,00)}"'] \arrow[dr, "{(01,01)}"] & \overset{(0,1)}{\bullet} \arrow[d, "{(01,11)}"] \\ \overset{(1,0)}{\bullet} \arrow[r, "{(11,01)}"'] & \overset{(1,1)}{\bullet} \end{tikzcd}$$

where the 2-simplex $(011,001)$ is the bottom left triangle, and $(001,011)$ is the top right triangle in the square.

Remark 1.46. Even though $(00,01)$ is non-degenerate in $\mathbb{D}^1 \times \mathbb{D}^1$, 00 in $\mathbb{D}^1_1$ on its own is not. This shows that a correct notion of product that is compatible with our geometric intuition requires the presence of degenerate simplices.

Lemma 1.47. *There is an isomorphism of simplicial sets $\partial\mathbb{D}^1 \cong \mathbb{D}^0 \coprod \mathbb{D}^0$.*

Proof. A map $\mathbb{D}^0 \coprod \mathbb{D}^0 \to \partial\mathbb{D}^1$ is uniquely given by a pair of maps $\mathbb{D}^0 \to \partial\mathbb{D}^1$. By the Yoneda lemma, each one of these maps corresponds to an element in $\partial\mathbb{D}^1_0 = \{0,1\}$. Hence, we have the map $0 \coprod 1 \colon \mathbb{D}^0 \coprod \mathbb{D}^0 \to \partial\mathbb{D}^1$, which by construction is an injection. Finally, $(\mathbb{D}^0 \coprod \mathbb{D}^0)_n = \{00...0\} \coprod \{00...0\}$ has two elements, and so the map is a bijection. $\square$

Lemma 1.48. *We have three isomorphisms of simplicial sets.*

- $\mathbb{L}^2_0 \cong \mathbb{D}^1 \coprod^{0,0}_{\mathbb{D}^0} \mathbb{D}^1$
- $\mathbb{L}^2_1 \cong \mathbb{D}^1 \coprod^{1,0}_{\mathbb{D}^0} \mathbb{D}^1$
- $\mathbb{L}^2_2 \cong \mathbb{D}^1 \coprod^{1,1}_{\mathbb{D}^0} \mathbb{D}^1$

Proof. We focus on the first case, for the other two see Exercise 1.10. The other cases are analogous. We start by constructing a map of simplicial sets $\mathbb{D}^1 \coprod^{0,0}_{\mathbb{D}^0} \mathbb{D}^1 \to \mathbb{L}^2_0$. First observe we have the following commutative diagram

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{D}^1 \arrow[d, "01"] \\ \mathbb{D}^1 \arrow[r, "02"] & \mathbb{L}^2_0 \end{tikzcd},$$

where we used the fact that $(\mathbb{L}^2_0)_1 = \{00,01,11,02,22\}$ and the Yoneda lemma (Lemma 1.26) to construct the map $\mathbb{D}^1 \to \mathbb{L}^2_0$. Here the commutativity follows from the fact that the image of $0 \in \mathbb{D}^0_0$ in both directions is $0 \in (\mathbb{L}^2_0)_0$.

So, there is a unique map $01+02\colon \mathbb{D}^1\coprod_{\mathbb{D}^0}\mathbb{D}^1 \to \mathbb{L}^2_0$. We now directly observe that at level k, the map $01+02$ is given by

$$\{00...0,...,11...1\}\coprod_{00...0}\{00...0,...,11...1\}\to\{00...0,11...1,00...02,...,22...2\}.$$

Concretely, the map is given by the identity on the first component and maps 00...111 to 00...222 on the second component, and is hence a bijection. $\square$

Lemma 1.49. *Let $n\geq 2$. There is an isomorphism of simplicial sets*

$$\mathbb{S}\mathrm{p}^n\cong\mathbb{D}^1\coprod_{\mathbb{D}^0}...\coprod_{\mathbb{D}^0}\mathbb{D}^1$$

Proof. Exercise 1.11 $\square$

Finally, let us conclude with a proposition that relates limits and colimits of simplicial sets to mapping simplicial sets.

Proposition 1.50. *Let $F\colon I\to s\mathcal{S}\mathrm{et}$ be a diagram of simplicial sets. Then for every simplicial set X, there are isomorphisms of simplicial sets*

$$\mathrm{Map}(\mathrm{colim}_{i\in I}F(i),X)\cong\lim_{i\in I}\mathrm{Map}(F(i),X),$$

$$\mathrm{Map}(X,\lim_{i\in I}F(i))\cong\lim_{i\in I}\mathrm{Map}(X,F(i)).$$

Proof. Exercise 1.12 $\square$

Exercises.

Exercise 1.1. Write down explicitly the maps

$$d^0,d^1,d^2\colon[1]\to[2]\qquad\text{and}\qquad s^0\colon[1]\to[0]$$

using the notation from Notation 1.2.

Exercise 1.2. Define what it means for an n-simplex of a simplicial set to be degenerate. For each of the simplices

$$001,\qquad 011,\qquad 012,\qquad 022$$

in $\mathbb{D}^2_2$, decide whether it is degenerate or non-degenerate.

Exercise 1.3. Prove that $\mathrm{Hom}_{\mathbb{\Delta}}([m],[n])$ has $\binom{m+n+1}{n}$ elements.

Exercise 1.4. Prove Lemma 1.20: the simplicial set $\mathbb{D}^0$ is terminal in $s\mathcal{S}\mathrm{et}$.

Exercise 1.5. Verify the following special cases of the cosimplicial identities:

$$d^2d^0=d^0d^1,\qquad s^1s^0=s^0s^2,\qquad s^1d^1=\mathrm{id},\qquad s^1d^3=d^2s^1.$$

Make sure to specify the relevant domains and codomains.

Exercise 1.6. Prove that the boundary $\partial\mathbb{D}^n$ is a sub-simplicial set of $\mathbb{D}^n$.

Exercise 1.7. Prove that the horn $\mathbb{L}^n_i$ is a sub-simplicial set of $\mathbb{D}^n$.

Exercise 1.8. Prove Lemma 1.41. That is, show that for $n\geq 2$ one has

$$\mathbb{S}\mathrm{p}^n_0=\mathbb{D}^n_0$$

Exercise 1.9. Prove Lemma 1.42. That is, show that for $n,k\geq 2$ every element in $\mathbb{S}\mathrm{p}^n_k$ is degenerate.

Exercise 1.10. Prove the second and third isomorphisms from Lemma 1.48, namely

$$\mathbb{L}^2_1\cong\mathbb{D}^1\coprod_{\mathbb{D}^0}^{1,0}\mathbb{D}^1\qquad\text{and}\qquad\mathbb{L}^2_2\cong\mathbb{D}^1\coprod_{\mathbb{D}^0}^{1,1}\mathbb{D}^1.$$

Exercise 1.11. Use induction and Lemma 1.48 to prove Lemma 1.49. That is, show that for $n \geq 2$ one has

$$\mathbb{S}\mathrm{p}^n \cong \mathbb{D}^1 \coprod_{\mathbb{D}^0} \dots \coprod_{\mathbb{D}^0} \mathbb{D}^1.$$

Exercise 1.12. Prove Proposition 1.50. That is, use the Yoneda lemma and the fact that limits and colimits commute with hom-sets to show that for a diagram $F\colon I \to s\mathcal{S}et$ of simplicial sets and every simplicial set X, there are isomorphisms of simplicial sets

$$\mathrm{Map}(\mathrm{colim}_{i\in I} F(i), X) \cong \lim_{i\in I} \mathrm{Map}(F(i), X),$$

$$\mathrm{Map}(X, \lim_{i\in I} F(i)) \cong \lim_{i\in I} \mathrm{Map}(X, F(i)).$$

Exercise 1.13. Using the Yoneda lemma, prove that

$$\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^n, \mathbb{D}^k) \cong \mathrm{Hom}_{\mathbb{\Delta}}([n], [k]).$$

Exercise 1.14. List all elements of $\mathrm{Hom}_{\mathbb{\Delta}}([1], [2])$. Which of these are injective maps?

Exercise 1.15. List all elements of $\mathrm{Hom}_{\mathbb{\Delta}}([2], [1])$. Which of these are surjective maps?

Exercise 1.16. Express the maps

$$001, \qquad 011 \in \mathrm{Hom}_{\mathbb{\Delta}}([2], [1])$$

in terms of the generating morphisms s^i.

Exercise 1.17. Compute the following face and degeneracy maps in $\mathbb{D}^2$:

$$d_0(012), \qquad d_1(012), \qquad d_2(012),$$

and

$$s_0(02), \qquad s_1(02).$$

Exercise 1.18. Compute the non-degenerate simplices of $\partial\mathbb{D}^3$. How many non-degenerate simplices does $\partial\mathbb{D}^3$ have in dimensions 0, 1, 2, and 3?

Exercise 1.19. Compute $(\mathbb{L}^2_0)_1$, $(\mathbb{L}^2_1)_1$, and $(\mathbb{L}^2_2)_1$. For each horn, identify which non-degenerate edge of $\partial\mathbb{D}^2$ is missing.

Exercise 1.20. Compute the non-degenerate simplices of $\mathbb{S}\mathrm{p}^4$. In particular, list its non-degenerate 0-simplices and 1-simplices, and explain why it has no non-degenerate simplices in dimension 2 or higher.

Exercise 1.21. Let $S = \{0, 2\} \subseteq \mathbb{D}^2_0$. Compute the sub-simplicial set $\bar{S}$ of $\mathbb{D}^2$ generated by S. Show that $\bar{S} \cong \mathbb{D}^1$.

Exercise 1.22. Identify the 1-simplices and 2-simplices of $\mathbb{D}^1 \times \mathbb{D}^1$ that are non-degenerate.

Exercise 1.23. Show that $\mathrm{Map}(\mathbb{D}^0, X) \cong X$ for every simplicial set X. Then compute $\mathrm{Map}(\mathbb{D}^0, \mathbb{D}^1)_2$ explicitly.

2. Simplicial Sets from Categories and Topological Spaces

In the previous section we introduced simplicial sets. We now want to see how we can use simplicial sets to study both categories and topological spaces.

2.1. **From Category Theory to Simplicial Sets.** We now start relating categories to simplicial sets. Here the key idea is the *nerve* of a category. For more technical introductions see [Rez22, Section 4] or [Lur26, Subsection 002M].

By definition of $\mathbb{\Delta}$ (Definition 1.1), there is a functor $\mathbb{\Delta} \to \mathcal{C}\mathrm{at}$, which is the identity on objects and morphisms. We now have the following definition.

Definition 2.1 (Nerve)**.** Let $N\colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ be the functor given by the restricted Yoneda embedding applied to the functor $\mathbb{\Delta} \to \mathcal{C}\mathrm{at}$.

Remark 2.2. More explicitly, we can describe the nerve as follows.

- $n = 0$: $N\mathcal{C}_0 = \mathrm{Fun}([0], \mathcal{C}) = \mathrm{Obj}_{\mathcal{C}}$ is the set of objects of $\mathcal{C}$.
- $n = 1$: $N\mathcal{C}_1 = \mathrm{Fun}([1], \mathcal{C}) = \mathrm{Mor}_{\mathcal{C}}$ is the set of morphisms of $\mathcal{C}$.
- $n \geq 2$: $N\mathcal{C}_n = \mathrm{Fun}([n], \mathcal{C})$ is the set of n composable morphisms in $\mathcal{C}$. In other words, an element of $N\mathcal{C}_n$ is a sequence of morphisms
$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} x_n$$
where $x_i \in \mathrm{Obj}_{\mathcal{C}}$ and $f_i\colon x_{i-1} \to x_i$.

Lemma 2.3. *Let $n \geq 0$. Then $N([n]) \cong \mathbb{D}^n$.*

Proof. Exercise 2.4 □

As a functor the nerve construction has very nice injectivity properties.

Proposition 2.4. *The nerve is fully faithful.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We need to show that the map
$$N\colon \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(N\mathcal{C}, N\mathcal{D})$$
is a bijection. The faithfulness of N is Exercise 2.5.

It remains to show that N is full. Given a map $f\colon N\mathcal{C} \to N\mathcal{D}$ of simplicial sets, we will define a functor $F\colon \mathcal{C} \to \mathcal{D}$, such that $N(F) = f$. We define F as follows.

- For an object $x \in \mathrm{Obj}_{\mathcal{C}} = (N\mathcal{C})_0$, define $F(x) := f_0(x)$.
- For a morphism $g\colon x \to y$ in $\mathcal{C}$, define $F(g) := f_1(g)$.
Here the simplicial relations
$$0^*(f_1(g)) = f_0(0^*(g)) = f_0(x) = F(x), \qquad 1^*(f_1(g)) = f_0(1^*(g)) = f_0(y) = F(y),$$
imply that if $g\colon x \to y$, then $F(g)\colon F(x) \to F(y)$.
- For a given object x in $\mathcal{C}$,
$$F(\mathrm{id}_x) = f_1(00^*(x)) = 00^*(f_0(x)) = 00^*(F(x)) = \mathrm{id}_{F(x)},$$
hence F preserves identities.
- For composable morphisms $x \xrightarrow{g} y \xrightarrow{h} z$ in $\mathcal{C}$, we have
$$F(h \circ g) = f_1(h \circ g) = f_1(02^*(g,h)) = 02^*(f_2(g,h)) = 02^*(f_1(g), f_1(h)) = f_1(h) \circ f_1(g) = F(h) \circ F(g),$$
hence F preserves composition.

Finally, we confirm that $N(F) = f$. By definition $N(F)_0 = f_0$ and $N(F)_1 = f_1$. More generally let $\sigma \in (N\mathcal{C})_n$ be an n-simplex, which, by Remark 2.2, is a string of n composable morphisms
$$\sigma = (x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} x_n).$$
Then the simplicial identities imply that
$$f_n(\sigma) = (f_1(g_1), \dots, f_1(g_n)) = (F(g_1), \dots, F(g_n)) = N(F)_n(\sigma).$$
This finishes the proof. □

This means we can see categories as special kinds of simplicial sets. But which simplicial sets do we get this way? Here we use the inclusion $i_n\colon \mathbb{S}\mathrm{p}^n \to \mathbb{D}^n$ (Definition 1.40) and the precomposition map.

Definition 2.5 (Segal condition)**.** A simplicial set X satisfies the *Segal condition* if the map

$$i_n^*\colon \mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^n, X) \to \mathrm{Hom}_{s\mathcal{S}et}(\mathcal{S}p^n, X)$$

is an isomorphism for $n \geq 2$.

Lemma 2.6. *Let X be a simplicial set and let $n \geq 2$. Then there is a bijection of sets*

$$\mathrm{Hom}_{s\mathcal{S}et}(\mathcal{S}p^n, X) \cong X_1 \times_{X_0}^{1^*,0^*} X_1 \times_{X_0}^{1^*,0^*} \ldots \times_{X_0}^{1^*,0^*} X_1.$$

Proof. We have the following chain of bijections:

$$\begin{aligned}
\mathrm{Hom}_{s\mathcal{S}et}(\mathcal{S}p^n, X) &\cong \mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^1 \coprod_{\mathbb{D}^0}^{1,0} \ldots \coprod_{\mathbb{D}^0}^{1,0} \mathbb{D}^1, X) && \text{Lemma 1.49} \\
&\cong \mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^1, X) \underset{\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^0,X)}{\times} \ldots \underset{\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^0,X)}{\times} \mathrm{Hom}_{s\mathcal{S}et}(\mathbb{D}^1, X) && \text{Colimit computation} \\
&\cong X_1 \times_{X_0}^{1^*,0^*} \ldots \times_{X_0}^{1^*,0^*} X_1 && \text{Lemma 1.26}
\end{aligned}$$

□

Proposition 2.7. *Let X be a simplicial set. Then the following are equivalent:*
(1) X satisfies the Segal condition.
(2) For every $n \geq 2$, and every map $f\colon \mathcal{S}p^n \to X$, the following diagram admits a unique lift

$$\begin{tikzcd} \mathcal{S}p^n \arrow[r, "f"] \arrow[d, hook, "i_n"'] & X \\ \mathbb{D}^n \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

Proof. Exercise 2.6 □

Proposition 2.8. *Let X be a simplicial set. Then the following are equivalent:*
(1) X satisfies the Segal condition.
(2) For $n \geq 2$, the composition map

$$X_n \cong \mathrm{Hom}(\mathbb{D}^n, X) \xrightarrow{i_n^*} \mathrm{Hom}(\mathcal{S}p^n, X) \cong X_1 \times_{X_0}^{1^*,0^*} X_1 \times_{X_0}^{1^*,0^*} \ldots \times_{X_0}^{1^*,0^*} X_1,$$

is a bijection.

Proof. Exercise 2.7 □

The Segal condition, by its definition, is an infinite condition. It is an interesting observation that we can turn it into a single condition via mapping simplicial sets. The proof is combinatorially involved and will be skipped for now.

Proposition 2.9. *A simplicial set X satisfies the Segal condition if and only if the map*

$$i_2^*\colon \mathrm{Map}(\mathbb{D}^2, X) \to \mathrm{Map}(\mathcal{S}p^2, X)$$

is an isomorphism of simplicial sets.

Intuition 2.10. As we saw in Proposition 2.7, $\mathrm{Hom}(\mathbb{D}^n, X) \to \mathrm{Hom}(\mathcal{S}p^n, X)$ being a bijection precisely corresponds to the existence of unique lifts along the map i_n. From that perspective, the statement that $\mathrm{Map}(\mathbb{D}^2, X) \to \mathrm{Map}(\mathcal{S}p^2, X)$ is an isomorphism can be understood as saying that each simplicial set of lifts is isomorphic to $\mathbb{D}^0$, which is the simplicial set analogue of a lifting condition.

Thus intuitively Proposition 2.9 is stating that by strengthening the lifting condition to a simplicial set of lifts, we can hence reduce the infinite Segal condition to a single map.

Remark 2.11. By Example 1.43, the Segal condition is also equivalent to the condition that the map

$$i_2^*\colon \mathrm{Map}(\mathbb{D}^2, X) \to \mathrm{Map}(\mathbb{L}_1^2, X)$$

is an isomorphism of simplicial sets.

Remark 2.12. As our foundation is set-theoretic, a "simplicial set of lifts" is not a primitive concept, in contrast to hom-sets and bijections. Thus formulating the condition in Proposition 2.9 required setting up relevant simplicial machinery. However, the result already gives a first glimpse how conditions can be simplified in an alternative foundation.

We now have the following important observation.

Proposition 2.13. *Let $\mathcal{C}$ be a category. Then the nerve $N\mathcal{C}$ satisfies the Segal condition.*

Proof. Exercise 2.8 □

Before we move on, this observation can help us compute the nerve in many new situations. Here we use the product of categories.

Proposition 2.14. *Let $\mathcal{C}, \mathcal{D}$ be two categories. Then $N(\mathcal{C} \times \mathcal{D}) \cong N\mathcal{C} \times N\mathcal{D}$.*

Proof. Exercise 2.9 □

What is fascinating is that the other direction of Proposition 2.13 also holds.

Theorem 2.15. *A simplicial set X is isomorphic to the nerve of a category if and only if X satisfies the Segal condition.*

Proof. We already proved one direction in Proposition 2.13. Let X be a simplicial set that satisfies the Segal condition. Using the Segal conditions, we fix the notation and denote the inverse Segal maps by $\mu_2 \colon X_1 \times_{X_0} X_1 \to X_2$ and $\mu_3 \colon X_1 \times_{X_0} X_1 \times_{X_0} X_1 \to X_3$. We construct a category $\mathcal{C}$ such that $N\mathcal{C} \cong X$ as follows.

- $\mathrm{Obj}_{\mathcal{C}} = X_0$.
- For two objects x, y in $\mathrm{Obj}_{\mathcal{C}}$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is given by the pullback

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & X_1 \arrow[d, "{(0^*,1^*)}"] \\ \{*\} \arrow[r, "{(x,y)}"] & X_0 \times X_0 \end{tikzcd} .$$

- The identity morphism on an object x is given by $00^*(x)$.
- For two morphisms $f \colon x \to y$ and $g \colon y \to z$, we define the composition as $g \circ f = 02^*\mu_2(f, g)$.
- For three morphisms f, g, h that are composable, we have

$$(h \circ g) \circ f = 02^*\mu_2(f, 02^*\mu_2(g, h)) = 03^*\mu_3(f, g, h) = 02^*\mu_2(02^*\mu_2(f, g), h) = h \circ (g \circ f).$$

  Hence, composition is associative.
- For a morphism $f \colon x \to y$, we have

$$f \circ \mathrm{id}_x = 02^*\mu_2(00^*(x), f) = f \text{ and } \mathrm{id}_y \circ f = 02^*\mu_2(f, 00^*(y)) = f.$$

  Hence, composition is unital.

Finally, we need to show that $N\mathcal{C} \cong X$. For $n = 0$, we have $N\mathcal{C}_0 = \mathrm{Obj}_{\mathcal{C}} = X_0$. For $n = 1$, we have $N\mathcal{C}_1 = \mathrm{Mor}_{\mathcal{C}} = X_1$. For $n \geq 2$, we have

$$N\mathcal{C}_n \cong N\mathcal{C}_1 \times^{1^*,0^*}_{N\mathcal{C}_0} \dots \times^{1^*,0^*}_{N\mathcal{C}_0} N\mathcal{C}_1 \cong X_1 \times^{1^*,0^*}_{X_0} \dots \times^{1^*,0^*}_{X_0} X_1 \cong X_n.$$

Hence, $N\mathcal{C} \cong X$ and we are done. □

Remark 2.16. As part of the construction we in particular observed that for two objects x, y in $\mathcal{C}$, we can reconstruct the hom-set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ via the nerve of $\mathcal{C}$ as the pullback

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & N\mathcal{C}_1 \arrow[d, "{(0^*,1^*)}"] \\ \{*\} \arrow[r, "{(x,y)}"] & N\mathcal{C}_0 \times N\mathcal{C}_0 \end{tikzcd} .$$

Combining these results we can conclude the following.

Definition 2.17. Let $s\mathcal{S}\mathrm{et}^{\mathcal{S}eg}$ be the category whose objects are simplicial sets satisfying the Segal condition and whose morphisms are maps of simplicial sets.

Corollary 2.18. *The nerve* $N\colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}^{\mathcal{S}eg}$ *is an equivalence of categories.*

Proof. Exercise 2.10 □

Thus we have shown that, up to equivalence, category theory is simply the study of a particular class of simplicial sets, namely those that satisfy the Segal condition.

2.2. **From Algebraic Topology to Simplicial Sets.** We now want to see how we can use simplicial sets to study the homotopy theory of topological spaces. This section primarily serves to understand the transition process from topological spaces to simplicial sets. Hence, we will review relevant facts from algebraic topology without proof, providing references when needed. Further details can of course be found in any standard textbook on algebraic topology, such as [Hat02].

Algebraic topology studies topological spaces via algebraic invariants. Prominent examples include the *homotopy groups*, which we review now.

Definition 2.19 (Homotopy)**.** Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. A *homotopy* from f to g is a continuous map $H\colon X \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. We say that f and g are *homotopic* if there exists a homotopy from f to g.

For this next definition recall pointed topological spaces.

Lemma 2.20 ([Hat02, Section 1.1])**.** *Let $(X, x), (Y, y)$ be two pointed topological spaces. Then the relation of being homotopic is an equivalence relation on* $\mathrm{Hom}_{\mathcal{T}op_*}((X, x), (Y, y))$.

Definition 2.21 (Homotopy groups)**.** Let (X, x) be a pointed topological space. For $n \geq 0$, the n-th homotopy set of X at x, denoted $\pi_n(X, x)$, is the set of homotopy classes of maps from the n-sphere S^n to X that send the base point of S^n to x.

Remark 2.22. Notice that $\pi_0(X, x)$, as defined in Definition 2.21, is independent of the choice of base point x, and is in fact the set of path-connected components of X. Hence, we will often write $\pi_0(X)$ instead of $\pi_0(X, x)$.

Remark 2.23. It is a classical result that $\pi_n(X, x)$ is a group for $n \geq 1$, and is abelian for $n \geq 2$ ([Hat02, Section 4.1]). Hence, following common convention, we will also adopt the term *homotopy group*. However, the group structure has no relevance for our purposes.

We can now define new invariants of topological spaces via homotopy groups.

Definition 2.24 (Weak homotopy equivalence)**.** A continuous map $f\colon X \to Y$ between topological spaces is a *weak homotopy equivalence* if it induces a bijection $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ and, for every base point $x \in X$, a bijection $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ for all $n \geq 1$.

One major question in algebraic topology is to determine whether two topological spaces are *weakly homotopy equivalent*, via some continuous map that is a weak homotopy equivalence. However, determining whether a given map is a weak homotopy equivalence is often impossibly difficult. One major breakthrough in this direction is the introduction of *CW-complexes*, and notably the *CW-approximation theorem*, as proven in [Hat02, Proposition 4.13].

Definition 2.25 (CW-complex)**.** A topological space X is a *CW-complex* if it is of the form $X = \bigcup_{n\geq 0} X_n$, where X_0 is a discrete topological space and, for $n \geq 1$, X_n is obtained from X_{n-1} by attaching n-cells $\partial D^n \to D^n$ along continuous maps $\partial D^n \to X_{n-1}$.

Theorem 2.26 (CW-approximation theorem)**.** *For every topological space X, there exists a CW-complex $\tilde{X}$ and a weak homotopy equivalence $\tilde{X} \to X$.*

This means that, if we only care about topological spaces up to weak homotopy equivalence, we can restrict our attention to CW-complexes. Fortunately, the inductive nature of CW-complexes allows us to effectively determine weak homotopy equivalences between them.

Definition 2.27 (Homotopy equivalence)**.** A continuous map $f\colon X \to Y$ between topological spaces is a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that both compositions $f \circ g$ and $g \circ f$ are homotopic to the identity.

We now have the following first lemma relating homotopy equivalences and weak homotopy equivalences (Definition 2.24). Concretely, we can rely on standard properties of homotopy groups, such as π_n being a functor and identifying homotopic maps, to show that homotopy equivalences are weak homotopy equivalences. See also [Hat02, Section 4.1].

Lemma 2.28. *Every homotopy equivalence is a weak homotopy equivalence.*

The converse does not hold in general. See [Hat02, Exercise 1.3.7] for a space that is weakly equivalent, but not homotopy equivalent, to a point. However, for CW-complexes, we have the following major result [Hat02, Theorem 4.5].

Theorem 2.29 (Whitehead's theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

This state of affairs naturally leads us to the following question: How can we effectively determine homotopy equivalences of CW-complexes? Here simplicial sets can provide a powerful approach.

Definition 2.30 (Topological n-simplex)**.** For $n \geq 0$, let $|\mathbb{D}^n|$ denote the topological n-simplex, which is the topological subspace of $\mathbb{R}^{n+1}$ given by

$$|\mathbb{D}^n| := \{(t_i)_{0\leq i\leq n} \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i\}.$$

Lemma 2.31. *The assignment* $[n] \mapsto |\mathbb{D}^n|$ *defines a functor* $|\mathbb{D}^\bullet|\colon \mathbb{\Delta} \to \mathcal{T}\mathrm{op}$.

Proof. We construct the functor on the morphisms d^i, s^i and then show the cosimplicial identities are satisfied (Lemma 1.5). Let $d^i\colon |\mathbb{D}^n| \to |\mathbb{D}^{n+1}|, s^i\colon |\mathbb{D}^n| \to |\mathbb{D}^{n-1}|$ be defined as follows:

$$d^i(t_0, ..., t_n) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_n)$$
$$s^i(t_0, ..., t_n) = (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_n).$$

We now move on to the simplicial identities. First, we have

$$s^i \circ d^i(t_0, ..., t_n) = s^i(t_0, ..., t_{i-1}, 0, t_i, ..., t_n) = (t_0, ..., t_{i-1}, t_i + 0, t_{i+1}, ..., t_n) = (t_0, ..., t_n).$$

For the remaining simplicial identities, see Exercise 2.17 □

Definition 2.32. Let $\mathrm{Sing}\colon \mathcal{T}\mathrm{op} \to s\mathcal{S}\mathrm{et}$ be the functor defined by applying the restricted Yoneda embedding to $|\mathbb{D}^\bullet|$.

Remark 2.33. More explicitly, for a given topological space X, we have the equality $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\mathbb{D}^n|, X)$.

Having defined Sing, we now want to understand how well it recovers the homotopy theory of topological spaces. For that we first develop a homotopy theory of simplicial sets, and then we show that Sing is compatible with it.

Definition 2.34. Two maps $f, g\colon X \to Y$ between simplicial sets are *homotopic* if there exists a map $H\colon X \times \mathbb{D}^1 \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$.

Definition 2.35 (Homotopy equivalence)**.** A map $f\colon X \to Y$ between simplicial sets is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that both compositions $f \circ g$ and $g \circ f$ are homotopic to the identity.

One early observation is that Sing is compatible with homotopies, which in particular will follow from explicit understanding of the functor Sing [GJ09, Chapter I].

Proposition 2.36. *The functor* Sing *sends homotopic maps to homotopic maps, and preserves homotopy equivalences.*

Thus we can study homotopy theory of topological spaces via simplicial sets. But do we really need all simplicial sets? Or can we restrict to a smaller class? Here we make the following observations.

Definition 2.37 (Kan complex)**.** A simplicial set K is a *Kan complex* if for every $n \geq 1$ and $0 \leq k \leq n$, every map $\mathbb{L}^n_k \to K$ extends to a map $\mathbb{D}^n \to K$

$$\begin{tikzcd} \mathbb{L}^n_k \arrow[r] \arrow[d] & K \\ \mathbb{D}^n \arrow[ur, dashed] \end{tikzcd},$$

or, equivalently, the map

$$\mathrm{Hom}(\mathbb{D}^n, K) \to \mathrm{Hom}(\mathbb{L}^n_k, K)$$

is surjective.

Proposition 2.38 ([GJ09, Lemma I.3.3])**.** *For every topological space X, the simplicial set $\mathrm{Sing}(X)$ is a Kan complex.*

Definition 2.39 (Category of Kan complexes)**.** Let $\mathcal{K}an$ be the category whose objects are Kan complexes and whose morphisms are maps of simplicial sets.

Proposition 2.38 shows that we have a functor $\mathrm{Sing}\colon \mathcal{T}op \to \mathcal{K}an$. Unlike the nerve functor, this functor is not an equivalence of categories. However, if we only consider homotopically relevant data, then things change. Let us see an example of that.

Definition 2.40 (Set of path-components)**.** Let S be a simplicial set. Let $\sim$ be the relation on S_0 given by $x \sim y$ if there exists an f in S_1, meaning a map $\mathbb{D}^1 \to S$, such that $0^*(f) = x$ and $1^*(f) = y$. Let $\pi_0(S) = S_0/\sim$, called the set of *path-components of* S, be the quotient of S_0 with respect to the equivalence relation generated by $\sim$.

For Kan complexes, Definition 2.40 behaves as one might expect, demonstrating the homotopical nature of Kan complexes.

Lemma 2.41. *Let K be a Kan complex. Then $\sim$ is already an equivalence relation on K_0.*

Proof. We need to show the relation is reflexive, symmetric, and transitive. Reflexivity is witnessed by the constant map $\mathbb{D}^1 \to K$. Now, let $H\colon \mathbb{D}^1 \to K$ be a homotopy from x to y. Let $\sigma\colon \mathbb{L}^2_0 \to K$ be the map sending 01 to H and 02 to id_x, where we used the description of $\mathbb{L}^2_0$ as a colimit (Lemma 1.48). As K is a Kan complex, there is a lift $H'\colon \mathbb{D}^2 \to K$. By the simplicial identities, $12^*H'\colon \mathbb{D}^1 \to K$ gives us the desired homotopy from y to x.

Similarly, if H_1 is a homotopy from x to y and H_2 is a homotopy from y to z, we can construct a homotopy from x to z by first constructing the map $H_1 + H_2\colon \mathbb{L}^2_1 \to K$ sending 01 to H_1 and 12 to H_2, then lifting it to a map $H_{12}\colon \mathbb{D}^2 \to K$, and finally restricting to $02^*H_{12}\colon \mathbb{D}^1 \to K$. □

Lemma 2.42. *Let X be a topological space. Then there is a bijection of sets $\pi_0(\mathrm{Sing}(X)) \cong \pi_0(X)$.*

Proof. By Remark 2.33, we have $\mathrm{Sing}(X)_0 = \mathrm{Hom}(|\mathbb{D}^0|, X)$ is the set of points of X, and $\mathrm{Sing}(X)_1 = \mathrm{Hom}(|\mathbb{D}^1|, X)$ is precisely the set of paths in X. Hence, by Definition 2.40, $\pi_0(\mathrm{Sing}(X))$ is the set of path-components of X, which is precisely $\pi_0(X)$. □

These results provide some first evidence that simplicial sets have a functioning homotopy theory and that Sing does indeed match the homotopy theory of topological spaces with Kan complexes. Ideally there should be a stronger result providing a precise match. While true, unfortunately, the general result is more involved, and we will only state it without proof.

Definition 2.43 (Homotopy category of CW-complexes)**.** Let $\mathrm{Ho}(\mathcal{T}op^{\mathrm{CW}})$ be the category with objects CW-complexes and morphisms homotopy classes of continuous maps.

Definition 2.44 (Homotopy category of Kan complexes)**.** Let $\mathrm{Ho}(\mathcal{K}an)$ be the category with objects Kan complexes and morphisms homotopy classes of maps.

Proposition 2.36 shows that there is an induced functor $\mathrm{Sing}\colon \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathrm{CW}}) \to \mathrm{Ho}(\mathcal{K}\mathrm{an})$. We now have the following major result.

Theorem 2.45 ([GJ09, Theorem 11.4])**.** *The functor* $\mathrm{Sing}\colon \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathrm{CW}}) \to \mathrm{Ho}(\mathcal{K}\mathrm{an})$ *is an equivalence of categories.*

The proof of this statement is a central result of simplicial homotopy theory. Proving it goes far beyond the scope of this introduction, and we refer the interested reader to [GJ09] for details.

Remark 2.46. One particular implication of these results is that homotopy equivalences of Kan complexes satisfy the famous 2-out-of-3 property, which states that if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of Kan complexes such that two of the three maps $f, g, g \circ f$ are homotopy equivalences, then so is the third. For a proof of this, we refer the reader to [GJ09, Proposition 9.2 CM2].

Exercises.

Exercise 2.1. Let $\mathcal{C}$ be a category, and let

$$\sigma = \left(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \right) \in N\mathcal{C}_3.$$

Compute the four faces $d_0(\sigma), d_1(\sigma), d_2(\sigma), d_3(\sigma)$ explicitly as 2-simplices of $N\mathcal{C}$.

Exercise 2.2. Let $f\colon x \to y$ be a 1-simplex of $N\mathcal{C}$. Compute the degenerate 2-simplices

$$s_0(f), \qquad s_1(f)$$

explicitly as strings of two composable morphisms in $\mathcal{C}$.

Exercise 2.3. Let

$$\tau = \left(x \xrightarrow{f} y \xrightarrow{g} z \right) \in N\mathcal{C}_2.$$

Compute the three restrictions

$$01^*(\tau), \qquad 12^*(\tau), \qquad 02^*(\tau)$$

as 1-simplices of $N\mathcal{C}$. Explain how the composition $g \circ f$ is encoded by the 2-simplex τ.

Exercise 2.4. Prove Lemma 2.3: for every $n \geq 0$, there is an isomorphism of simplicial sets

$$N([n]) \cong \mathbb{D}^n.$$

Exercise 2.5. Prove the faithful part of Proposition 2.4. That is, show that if $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors and

$$N(F) = N(G),$$

then $F = G$.

Exercise 2.6. Prove Proposition 2.7. That is, show that X satisfies the Segal condition if and only if every map

$$\mathcal{S}\mathrm{p}^n \to X$$

admits a unique extension along $i_n\colon \mathcal{S}\mathrm{p}^n \to \mathbb{D}^n$ for every $n \geq 2$.

Exercise 2.7. Prove Proposition 2.8. That is, use Lemmas 1.26 and 2.6 to show that the Segal condition is equivalent to the statement that, for every $n \geq 2$, the map

$$X_n \to X_1 \times_{X_0}^{1^*,0^*} X_1 \times_{X_0}^{1^*,0^*} \cdots \times_{X_0}^{1^*,0^*} X_1$$

is a bijection.

Exercise 2.8. Prove Proposition 2.13: for every category $\mathcal{C}$, the nerve $N\mathcal{C}$ satisfies the Segal condition.

Exercise 2.9. Prove Proposition 2.14 directly. That is, show that for every $n \geq 0$ there is a natural bijection

$$N(\mathcal{C} \times \mathcal{D})_n \cong (N\mathcal{C} \times N\mathcal{D})_n,$$

and check that these bijections are compatible with the face and degeneracy maps.

Exercise 2.10. Prove Corollary 2.18: the nerve functor

$$N\colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}^{Seg}$$

is an equivalence of categories.

Exercise 2.11. Let $\mathcal{C}$ be the discrete category associated to a set S. Compute $N\mathcal{C}_n$ for every $n \geq 0$. Show that $N\mathcal{C}$ is the constant simplicial set associated to S.

Exercise 2.12. Let M be a monoid, regarded as a category with one object $*$ and endomorphism monoid M. Compute NM_0, NM_1, NM_2, and NM_3. Then describe the face maps

$$d_i\colon NM_2 \to NM_1$$

and

$$d_i\colon NM_3 \to NM_2$$

in terms of the multiplication in M.

Exercise 2.13. Let G be a group, regarded as a category with one object. Let $(g,h) \in NG_2$ be the 2-simplex corresponding to the pair of composable morphisms

$$* \xrightarrow{g} * \xrightarrow{h} *.$$

Compute

$$01^*(g,h), \qquad 12^*(g,h), \qquad 02^*(g,h).$$

Then compute the four faces of a 3-simplex $(g,h,k) \in NG_3$.

Exercise 2.14. Compute the nerve of the product category $[1] \times [1]$ in dimensions 0, 1, and 2. Identify the non-degenerate 0-, 1-, and 2-simplices, and compare your answer with the simplicial set $\mathbb{D}^1 \times \mathbb{D}^1$.

Exercise 2.15. Let $\mathcal{C}$ be the category

$$x \xrightarrow{f} y \xrightarrow{g} z$$

with the composite $g \circ f\colon x \to z$. Compute $N\mathcal{C}_0$, $N\mathcal{C}_1$, and $N\mathcal{C}_2$. Which 2-simplices are degenerate, and which are non-degenerate?

Exercise 2.16. Let X be a simplicial set satisfying the Segal condition, and suppose $X_0 = \{x\}$ has exactly one element. Explain why the set X_1 carries a natural monoid structure. Then describe the nerve of the one-object category associated to this monoid.

Exercise 2.17. Prove that $|-|\colon \mathbb{\Delta} \to \mathcal{T}\mathrm{op}$ satisfies the remaining cosimplicial identities. That is,

(1) $d^j d^i = d^i d^{j-1}$ for $i < j$.
(2) $s^j s^i = s^i s^{j+1}$ for $i \leq j$.
(3) $s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ \mathrm{id} & i = j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$

Exercise 2.18. Compute the topological simplices

$$|\mathbb{D}^0|, \qquad |\mathbb{D}^1|, \qquad |\mathbb{D}^2|$$

explicitly as subspaces of Euclidean space. Then write down the maps

$$d^0, d^1\colon |\mathbb{D}^0| \to |\mathbb{D}^1|$$

and

$$s^0\colon |\mathbb{D}^1| \to |\mathbb{D}^0|$$

using the formulas from Lemma 2.31.

Exercise 2.19. Let S be a discrete topological space. Compute $\mathrm{Sing}(S)_n$ for every $n \geq 0$, and show that $\mathrm{Sing}(S)$ is the constant simplicial set associated to the underlying set of S.

3. Category Theory & Homotopy Theory: Two Paths towards Simplicial Spaces

In the previous section we studied two different aspects of the category of simplicial sets. We showed that one subcategory, namely $s\mathcal{S}\mathrm{et}^{\mathcal{S}eg}$ is equivalent to $\mathcal{C}\mathrm{at}$ and another subcategory, namely $\mathcal{K}\mathrm{an}$ has an equivalent homotopy theory to that of $\mathcal{T}\mathrm{op}$. A higher category should combine the categorical aspects of $\mathcal{C}\mathrm{at}$ and the homotopical aspects of $\mathcal{T}\mathrm{op}$. In particular both categories and topological spaces should be examples of higher categories. However, these two subcategories only intersect very trivially.

Lemma 3.1. *Let $\mathcal{C}$ be a category. Then $N\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.*

Proof. Let $\mathcal{C}$ be a category such that $N\mathcal{C}$ is a Kan complex. Let $f\colon x \to y$ be a morphism in $\mathcal{C}$. We need to show that f is an isomorphism. We now have the following bijections:

$$\begin{aligned}
\mathrm{Hom}(\mathbb{L}^2_0, N\mathcal{C}) &\cong \mathrm{Hom}(\mathbb{D}^1 \coprod_{\mathbb{D}^0}^{0,0} \mathbb{D}^1, N\mathcal{C}) && \text{Lemma 1.48} \\
&\cong \mathrm{Hom}(\mathbb{D}^1, N\mathcal{C}) \times^{0^*,0^*}_{\mathrm{Hom}(\mathbb{D}^0, N\mathcal{C})} \mathrm{Hom}(\mathbb{D}^1, N\mathcal{C}) && \text{Colimit computation} \\
&\cong N\mathcal{C}_1 \times^{0^*,0^*}_{N\mathcal{C}_0} N\mathcal{C}_1 && \text{Lemma 1.26} \\
&\cong \mathrm{Mor}_{\mathcal{C}} \times^{0^*,0^*}_{\mathrm{Obj}_{\mathcal{C}}} \mathrm{Mor}_{\mathcal{C}} && \text{Remark 2.2}
\end{aligned}$$

So, the surjectivity of the Kan lifting condition in Definition 2.37 means that for every pair of morphisms (f, g) with the same source, there exists a pair (f, h), such that $h \circ f = g$. Applying this to the pair (f, id_x), we get a left inverse h of f.

Now, repeating the same argument with $\mathbb{L}^2_2$, we similarly deduce the existence of a right inverse of f. Hence, f is an isomorphism and $\mathcal{C}$ is a groupoid.

The other direction is much more involved and so we refer the reader to [Lur26, Proposition 0037]. □

So, how can we combine two different types of objects which intersect almost trivially in a larger setting? One solution is to have two simplicial directions.

Definition 3.2 (Simplicial space)**.** A *simplicial space* is a functor $X\colon \mathbb{\Delta}^{\mathrm{op}} \to s\mathcal{S}\mathrm{et}$. The category of simplicial spaces is the functor category $s\mathcal{S} := \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, s\mathcal{S}\mathrm{et})$.

Remark 3.3. We have an isomorphism of categories

$$\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, s\mathcal{S}\mathrm{et}) = \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})) \cong \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Thus we have two equivalent ways to depict a simplicial space X:

$$
\begin{tikzcd}
X_{00} \arrow[r] \arrow[d] & X_{10} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & X_{20} \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[d] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] \\
X_{01} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & X_{11} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & X_{21} \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[d, shift left=2] \arrow[d, shift right=2] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] \\
X_{02} \arrow[u, shift left=4] \arrow[u] \arrow[u, shift right=4] \arrow[r] \arrow[d, shift left=4] \arrow[d] \arrow[d, shift right=4] & X_{12} \arrow[u, shift left=4] \arrow[u] \arrow[u, shift right=4] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=4] \arrow[d] \arrow[d, shift right=4] & X_{22} \arrow[u, shift left=4] \arrow[u] \arrow[u, shift right=4] \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] \arrow[d, shift left=4] \arrow[d] \arrow[d, shift right=4] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] \\
\vdots \arrow[u, shift left=6] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[u, shift right=6] \arrow[d, leftrightsquigarrow] & \vdots \arrow[u, shift left=6] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[u, shift right=6] \arrow[d, leftrightsquigarrow] & \vdots \arrow[u, shift left=6] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[u, shift right=6] \arrow[d, leftrightsquigarrow] & \ddots \arrow[d, leftrightsquigarrow] \\
X_{0\bullet} \arrow[r] & X_{1\bullet} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & X_{2\bullet} \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6]
\end{tikzcd}
$$

Here X_{nm} are sets, whereas $X_{n\bullet}$ are simplicial sets.

As we now have two simplicial directions, we need two ways to include simplicial sets into simplicial spaces, giving us a categorical and spatial direction.

Definition 3.4 (Categorical inclusion)**.** The *categorical inclusion*, denoted $\mathrm{disc}\colon s\mathcal{S}\mathrm{et} \to s\mathcal{S}$, is the functor $\pi_1^*\colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induced by the projection $\pi_1\colon \mathbb{\Delta} \times \mathbb{\Delta} \to \mathbb{\Delta}$ onto the first factor.

Intuition 3.5. Given a simplicial set S, the diagram in Remark 3.3 of $\mathrm{disc}S$ is the simplicial space with $(\mathrm{disc}S)_{nk} = S_n$ and all vertical maps are identities.

Definition 3.6 (Spatial inclusion)**.** The *spatial inclusion*, denoted $\mathrm{const}\colon s\mathcal{S}\mathrm{et} \to s\mathcal{S}$, is the functor $\pi_2^*\colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induced by the projection $\pi_2\colon \mathbb{\Delta} \times \mathbb{\Delta} \to \mathbb{\Delta}$ onto the second factor.

Intuition 3.7. Given a simplicial set S, the diagram in Remark 3.3 of $\mathrm{const}S$ is the simplicial space with $(\mathrm{const}S)_{nk} = S_k$ and all horizontal maps are identities.

Remark 3.8. Based on Intuitions 3.5 and 3.7, disc is also called the *horizontal embedding* (as only the horizontal direction varies) and const the *vertical embedding* (as only the vertical direction varies).

Remark 3.9. For now the names categorical and spatial inclusion are just suggestive, but we will see later that the naming conventions are indeed justified.

Notation 3.10. Recall $\mathbb{D}^n$ (Definition 1.16), $\partial\mathbb{D}^n$ (Definition 1.33), $\mathbb{L}^n_k$ (Definition 1.37), and $\mathcal{S}\mathrm{p}^n$ (Definition 1.40). We adopt the notational conventions:

- $\Delta^n := \mathrm{disc}(\mathbb{D}^n)$, $\partial\Delta^n := \mathrm{disc}(\partial\mathbb{D}^n)$, $\Lambda^n_k := \mathrm{disc}(\mathbb{L}^n_k)$, $\mathrm{Sp}^n := \mathrm{disc}(\mathcal{S}\mathrm{p}^n)$ (for $n \geq 2$).
- $\mathbb{D}^l := \mathrm{const}(\mathbb{D}^l)$, $\partial\mathbb{D}^l := \mathrm{const}(\partial\mathbb{D}^l)$ (here we are identifying $\mathbb{D}^l$ with its image under const to simplify notation).
- More generally, for a simplicial set S, we will again denote the simplicial space $\mathrm{const}S$ by S, and note this is consistent with the previous item.

We saw before that we can use the Yoneda lemma to characterize maps out of $\mathbb{D}^n$. Now via the Yoneda lemma, we have the following analogous result.

Lemma 3.11. *Let X be a simplicial space and let $n, l \geq 0$. Then there are natural bijections*

$$\mathrm{Hom}_{s\mathcal{S}}(\Delta^n \times \mathbb{D}^l, X) \cong X_{nl}.$$

Proof. Exercise 3.10 □

Analogous to simplicial sets, simplicial spaces have notions of mapping objects. However, in this case we have mapping simplicial spaces and mapping simplicial sets.

Definition 3.12 (Mapping simplicial space in $s\mathcal{S}$)**.** Let X, Y be two simplicial spaces. The *mapping simplicial space* $[\![X, Y]\!]$ is defined as the simplicial space

$$\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \xrightarrow{\mathcal{Y}\mathrm{on}^{\mathrm{op}}} s\mathcal{S}^{\mathrm{op}} \xrightarrow{(X\times -)^{\mathrm{op}}} s\mathcal{S}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,Y)} \mathcal{S}\mathrm{et}.$$

Moreover, $\mathrm{Map}(X, Y)$ is the simplicial set defined as $\mathrm{Map}(X, Y) := [\![X, Y]\!]_0$.

Analogous to Remark 1.29, the (k, n)-simplices of $[\![X, Y]\!]$ are maps $X \times \Delta^k \times \mathbb{D}^n \to Y$, and the face and degeneracy maps are induced by the corresponding maps of $\Delta^k \times \mathbb{D}^n$. In particular, the vertices of $[\![X, Y]\!]$ are precisely the maps from X to Y.

Remark 3.13. By Notation 3.10, if X, Y are simplicial sets, then $\mathrm{Map}(X, Y) = \mathrm{Map}(\mathrm{const}X, \mathrm{const}Y)$, meaning the notions of mapping simplicial sets coincide.

We now make several important observations about Map that mirror the behavior of Hom. First we have a generalized form of the Yoneda lemma.

Lemma 3.14. *Let X be a simplicial space and let $n \geq 0$. Then there is a natural isomorphism of simplicial sets*

$$\mathrm{Map}(\Delta^n, X) \cong X_n.$$

Proof. Exercise 3.12 □

Beyond maps out of Δ^n, which we can evaluate via the Yoneda lemma, there are two other cases we can easily compute.

Lemma 3.15. *Let X be a simplicial space, and $\varnothing$ denote the empty simplicial space (all levels are the empty set). Then, $[\![\varnothing, X]\!] \cong \Delta^0$ and so $\mathrm{Map}(\varnothing, X) \cong \mathbb{D}^0$.*

Proof. Exercise 3.13 □

Lemma 3.16. *Let X be a simplicial space. Then, $[\![X, \Delta^0]\!] \cong \Delta^0$ and so $\mathrm{Map}(X, \Delta^0) \cong \mathbb{D}^0$.*

Proof. Exercise 3.14 □

By construction, the mapping space is suitably functorial and concretely we have the following observation.

Lemma 3.17. *Let $f\colon X \to Y$ be a map of simplicial spaces and let Z be a simplicial space. The collection of maps $(f^*)_{k,n}\colon [\![Y, Z]\!]_{k,n} \to [\![X, Z]\!]_{k,n}$, given by*

$$(\sigma\colon Y \times \Delta^k \times \mathbb{D}^n \to Z) \mapsto (\sigma \circ (f \times \mathrm{id}_{\Delta^k \times \mathbb{D}^n})\colon X \times \Delta^k \times \mathbb{D}^n \to Z),$$

defines a map of simplicial spaces $f^*\colon [\![Y, Z]\!] \to [\![X, Z]\!]$.

Proof. Exercise 3.15 □

Remark 3.18. f^* restricts to a map of simplicial sets $(f^*)_0\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$, which we also denote by f^* throughout.

Similarly, we have the following analogous lemma.

Lemma 3.19. *Let $f\colon X \to Y$ be a map of simplicial spaces and let Z be a simplicial space. The collection of maps $(f_*)_{k,n}\colon [\![Z, X]\!]_{k,n} \to [\![Z, Y]\!]_{k,n}$, given by*

$$(\sigma\colon Z \times \Delta^k \times \mathbb{D}^n \to X) \mapsto (f \circ \sigma\colon Z \times \Delta^k \times \mathbb{D}^n \to Y),$$

defines a map of simplicial spaces $f_*\colon [\![Z, X]\!] \to [\![Z, Y]\!]$.

Proof. Exercise 3.16 □

We now move on to colimits and limits in $s\mathcal{S}$. As a functor category colimits and limits in $s\mathcal{S}$ are computed pointwise, however, we have the following generalized version of the universal property of colimits and limits for mapping spaces.

Proposition 3.20. *Let $F\colon I \to s\mathcal{S}$ be a diagram of simplicial spaces. Then for every simplicial space X, there are isomorphisms of simplicial spaces*

$$[\![\mathrm{colim}_{i\in I} F(i), X]\!] \cong \lim_{i\in I} [\![F(i), X]\!],$$
$$[\![X, \lim_{i\in I} F(i)]\!] \cong \lim_{i\in I} [\![X, F(i)]\!],$$

and there are analogous statements for the mapping simplicial sets Map.

Proof. Exercise 3.17 □

Let us see some examples.

Example 3.21. Let X be a simplicial space. Then, combining Proposition 3.20 and Lemma 3.14, we get

$$\mathrm{Map}(\Delta^0 \coprod ... \coprod \Delta^0, X) \cong \mathrm{Map}(\Delta^0, X) \times ... \times \mathrm{Map}(\Delta^0, X) \cong X_0 \times ... \times X_0.$$

Example 3.22. In particular, analogous to Lemma 1.47, we have $\partial\Delta^1 \cong \Delta^0 \coprod \Delta^0$, and so for any simplicial space X, we have $\mathrm{Map}(\partial\Delta^1, X) \cong X_0 \times X_0$.

A general simplicial space is evidently not a suitable model for a higher category, as there is not even a notion of composition. We need to add some conditions.

Exercises.

Exercise 3.1. Let S be a simplicial set. Compute

$$(\mathrm{disc} S)_{nk} \qquad \text{and} \qquad (\mathrm{const} S)_{nk}$$

for all $n, k \geq 0$. Then describe which simplicial direction is non-trivial in each case.

Exercise 3.2. Let $S = \mathbb{D}^1$. Compute the four sets

$$(\mathrm{disc}\mathbb{D}^1)_{00}, \quad (\mathrm{disc}\mathbb{D}^1)_{10}, \quad (\mathrm{disc}\mathbb{D}^1)_{01}, \quad (\mathrm{disc}\mathbb{D}^1)_{11},$$

and compare them with

$$(\mathrm{const}\mathbb{D}^1)_{00}, \quad (\mathrm{const}\mathbb{D}^1)_{10}, \quad (\mathrm{const}\mathbb{D}^1)_{01}, \quad (\mathrm{const}\mathbb{D}^1)_{11}.$$

Exercise 3.3. Using Notation 3.10, write explicit formulas for

$$(\Delta^n)_{kl}, \qquad (\partial\Delta^n)_{kl}, \qquad (\mathrm{Sp}^n)_{kl}, \qquad (\mathbb{D}^l)_{kn}$$

in terms of ordinary simplicial sets.

Exercise 3.4. Let X, Y be simplicial spaces. Compute

$$\mathrm{Map}(X, Y)_0 \qquad \text{and} \qquad \mathrm{Map}(X, Y)_1$$

directly from the definition.

Exercise 3.5. Let $\mathcal{C}$ be a category. Using Lemma 1.48, identify

$$\mathrm{Hom}_{s\mathcal{S}et}(\mathbb{L}^2_0, N\mathcal{C})$$

with pairs of morphisms in $\mathcal{C}$ having the same source.

Exercise 3.6. Let $\mathcal{C} = [1]$ be the category with two objects $0, 1$ and a single non-identity morphism $0 \to 1$. Construct explicitly a map

$$\mathbb{L}^2_0 \to N[1]$$

which admits no extension to a map

$$\mathbb{D}^2 \to N[1].$$

Conclude that $N[1]$ is not a Kan complex.

Exercise 3.7. Prove that the functor
$$\mathrm{disc}\colon s\mathcal{S}\mathrm{et} \to s\mathcal{S}$$
is fully faithful.

Exercise 3.8. Prove that the functor
$$\mathrm{const}\colon s\mathcal{S}\mathrm{et} \to s\mathcal{S}$$
is fully faithful.

Exercise 3.9. Prove that if S is a constant simplicial set, then
$$\mathrm{disc} S \cong \mathrm{const} S$$
as simplicial spaces.

Exercise 3.10. Prove Lemma 3.11. That is, show that for every simplicial space X and every $n, l \geq 0$, there is a natural bijection
$$\mathrm{Hom}_{s\mathcal{S}}(\Delta^n \times \mathbb{D}^l, X) \cong X_{nl}.$$

Exercise 3.11. Deduce from Lemma 3.11 that
$$\mathrm{Hom}_{s\mathcal{S}}(\Delta^n, X) \cong X_{n0}$$
for every simplicial space X and every $n \geq 0$.

Exercise 3.12. Prove Lemma 3.14. That is, prove that for every simplicial space X and every $n \geq 0$, there is a natural isomorphism of simplicial sets
$$\mathrm{Map}(\Delta^n, X) \cong X_n.$$

Exercise 3.13. Prove Lemma 3.15. That is, prove that for every simplicial space X,
$$[\![\emptyset, X]\!] \cong \Delta^0$$
and hence
$$\mathrm{Map}(\emptyset, X) \cong \mathbb{D}^0.$$

Exercise 3.14. Prove Lemma 3.16. That is, prove that for every simplicial space X,
$$[\![X, \Delta^0]\!] \cong \Delta^0$$
and hence
$$\mathrm{Map}(X, \Delta^0) \cong \mathbb{D}^0.$$

Exercise 3.15. Prove Lemma 3.17. That is, let $f\colon X \to Y$ be a map of simplicial spaces and let Z be a simplicial space. Prove that precomposition with f defines a map of simplicial spaces
$$f^*\colon [\![Y, Z]\!] \to [\![X, Z]\!].$$
Then describe the induced map of simplicial sets
$$f^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z).$$

Exercise 3.16. Prove Lemma 3.19. That is, let $f\colon X \to Y$ be a map of simplicial spaces and let Z be a simplicial space. Prove that postcomposition with f defines a map of simplicial spaces
$$f_*\colon [\![Z, X]\!] \to [\![Z, Y]\!].$$
Then describe the induced map of simplicial sets
$$f_*\colon \mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y).$$

Exercise 3.17. Prove Proposition 3.20 by evaluating both sides on (k, n)-simplices.

Exercise 3.18. Prove that
$$\partial\Delta^1 \cong \Delta^0 \coprod \Delta^0.$$
Then use this to prove that for every simplicial space X,
$$\mathrm{Map}(\partial\Delta^1, X) \cong X_0 \times X_0.$$

Exercise 3.19. Compute the bidegrees
$$(\Delta^2)_{00}, \qquad (\Delta^2)_{10}, \qquad (\Delta^2)_{01}, \qquad (\Delta^2)_{11}.$$
Then compute the same four bidegrees for the spatial simplex $\mathbb{D}^2$.

Exercise 3.20. Compute
$$(\partial\Delta^1)_{00}, \qquad (\partial\Delta^1)_{10}, \qquad (\partial\Delta^1)_{01}, \qquad (\partial\Delta^1)_{11}.$$
Compare your answer with $\Delta^0 \coprod \Delta^0$.

Exercise 3.21. Compute
$$(\mathrm{Sp}^2)_{00}, \qquad (\mathrm{Sp}^2)_{10}, \qquad (\mathrm{Sp}^2)_{20}.$$
Then compare Sp^2 with Δ^2 and $\partial\Delta^2$ in these bidegrees.

Exercise 3.22. Let X be a simplicial space. Prove that $[\![\Delta^0, X]\!] \cong X$ and hence $\mathrm{Map}(\Delta^0, X) \cong X_0$.

Exercise 3.23. Let X be a simplicial space. Compute
$$\mathrm{Map}(\partial\Delta^1, X)_k$$
for arbitrary $k \geq 0$. Explain why this gives an isomorphism of simplicial sets
$$\mathrm{Map}(\partial\Delta^1, X) \cong X_0 \times X_0.$$

Exercise 3.24. Compute
$$\mathrm{Map}(\Delta^1, \Delta^2).$$
In particular, identify its set of 0-simplices and decide whether this mapping simplicial set is constant.

Exercise 3.25. Compute
$$\mathrm{Map}(\Delta^1, \mathbb{D}^2).$$
Here $\mathbb{D}^2$ is regarded as a simplicial space via the spatial inclusion. Compare your answer with the previous exercise.

Exercise 3.26. Compute
$$\mathrm{Hom}_{s\mathcal{S}}(\Delta^1 \times \mathbb{D}^2, \Delta^3 \times \mathbb{D}^1)$$
using Lemma 3.11. Then compute the cardinality of this set.

Exercise 3.27. Let S be a simplicial set and $n \geq 0$. Compute
$$\mathrm{Map}(\Delta^n, \mathrm{disc} S)$$
as a simplicial set. Then compute
$$\mathrm{Map}(\Delta^n, \mathrm{const} S).$$
Explain how the answers reflect the difference between the categorical and spatial inclusions.

Exercise 3.28. Let X be a simplicial space. Compute
$$\mathrm{Map}(\Delta^0 \coprod \Delta^0 \coprod \Delta^0, X)$$
in terms of X_0.

Exercise 3.29. Let G be a group, regarded as a category with one object. Let
$$a, b \in G.$$
Interpreting a and b as edges in NG, compute the missing edge in a filler for a horn of the form
$$\mathbb{L}^2_0 \to NG$$
whose 01-edge is the element $a \in G$ and whose 02-edge is the element $b \in G$.

Exercise 3.30. Let G be a group, regarded as a category with one object. Compute the missing edge in a filler for a horn of the form
$$\mathbb{L}^2_2 \to NG$$
whose 02-edge is the element $b \in G$ and whose 12-edge is the element $c \in G$.

4. Interlude: Kan Fibrations as Dependent Spaces

In Subsection 2.2 we discussed Kan complexes as an analogue of topological spaces in the realm of simplicial sets. However, in the set-theoretic context we also use "dependent sets", otherwise known as functions. For example, we have the following result.

Lemma 4.1. *Assume we have the following commutative diagram of sets*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd} .$$

Then f is a bijection if and only if for every $x \in X$ the induced map on pre-images $f_x \colon p^{-1}(x) \to q^{-1}(x)$ is a bijection.

Proof. Exercise 4.5 □

Intuitively this result is stating that a "global bijection" between the sets Y and Z that sits over X is determined by "local bijections" on the fibers. Ideally, in the context of spaces we should have similar statements, where bijection is replaced by homotopy equivalence. However, unfortunately, this will fail without some assumption. Here we use the groupoid $I(1)$, which has two objects and a unique isomorphism between any two objects.

Lemma 4.2. *The map $N(0) \colon N[0] \to N(I(1))$ is a homotopy equivalence of Kan complexes.*

Proof. Following Lemma 3.1, $\mathbb{D}^0 = N[0]$ and $N(I(1))$ are both Kan complexes. Let $t \colon I(1) \to [0]$ be the unique functor. By uniqueness, we must have $t \circ 0 = \mathrm{id}_{[0]}$. Moreover, let $H \colon [1] \times I(1) \to I(1)$ be the functor defined as follows:

$$H(0,0) = 0, H(1,0) = 0, H(0,1) = 0, H(1,1) = 1,$$

and for morphisms the choices are unique. Now, by Proposition 2.14, $N([1] \times I(1)) \cong N([1]) \times N(I(1)) = \mathbb{D}^1 \times N(I(1))$. Then $NH \colon \mathbb{D}^1 \times N(I(1)) \to N(I(1))$ is a homotopy between $0 \circ t$ and $\mathrm{id}_{N(I(1))}$, as $NH|_{\{0\} \times N(I(1))} = N(0 \circ t)$ and $NH|_{\{1\} \times N(I(1))} = N(\mathrm{id}_{I(1)})$ (Exercise 4.6). □

Example 4.3. The functor $0 \colon [0] \to I(1)$ induces a commutative diagram of Kan complexes

$$\begin{tikzcd} \mathbb{D}^0 \arrow[rr, "N(0)"] \arrow[dr, "N(0)"'] & & N(I(1)) \arrow[dl, "\mathrm{id}"] \\ & N(I(1)) & \end{tikzcd} .$$

By Lemma 4.2, the top map is a homotopy equivalence. However, the pre-image over 1 is the map $\emptyset \to \mathbb{D}^0$ (Exercise 4.7), which is not a homotopy equivalence, and in fact admits no map in the other direction.

This is not the only situation where a suitable notion of dependency is needed. Here is another simple example.

Lemma 4.4. *Let $n \geq 2$. Assume we have the following commutative diagram of sets*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"', "\cong"] & A_{1,2} \arrow[d, "f_{1,2}"', "\cong"] & \cdots \arrow[l] \arrow[r] & A_{n-1,n} \arrow[d, "f_{n-1,n}"', "\cong"] & A_n \arrow[l] \arrow[d, "f_n"', "\cong"] \\ B_1 \arrow[r] & B_{1,2} & \cdots \arrow[l] \arrow[r] & B_{n-1,n} & B_n \arrow[l] \end{tikzcd} ,$$

such that all vertical maps are bijections. Then the induced map on limits is a bijection.

Proof. See Exercise 4.8 for the case $n = 2$. The general case follows by induction. □

Again, we would like to have a similar statement in the context of Kan complexes, where bijections are replaced by homotopy equivalences. However, once more this will fail without some assumption.

Example 4.5. Take the diagram of Kan complexes, where, by Lemma 4.2, all vertical maps are homotopy equivalences,

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "0"] \arrow[d, "="] & N(I(1)) \arrow[d, "\simeq"] & \mathbb{D}^0 \arrow[l, "1"'] \arrow[d, "="] \\ \mathbb{D}^0 \arrow[r] & \mathbb{D}^0 & \mathbb{D}^0 \arrow[l] \end{tikzcd}$$

For the top pullback, we have $(\mathbb{D}^0 \times^{0,1}_{N(I(1))} \mathbb{D}^0)_0 = \emptyset$ (Exercise 4.9). So, the induced map on pullbacks is the map $\emptyset \to \mathbb{D}^0$, which again is not a homotopy equivalence.

We hence need a proper notion of "dependent spaces" or "fibrations". Fortunately, there is no need to reinvent the wheel, and we can use the well-established notion of Kan fibrations.

Definition 4.6 (Kan fibration)**.** A map of simplicial sets $Y \to X$ is a *Kan fibration* if for every $n \geq 1, 0 \leq i \leq n$ the following diagram lifts

$$\begin{tikzcd} \mathbb{L}^n_i \arrow[r] \arrow[d] & Y \arrow[d, two heads] \\ \mathbb{D}^n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Notation 4.7. In diagrams we denote Kan fibrations by $\twoheadrightarrow$.

Remark 4.8. Kan fibrations are indeed relative versions of Kan complexes, as Kan fibrations $Y \to \mathbb{D}^0$ precisely correspond to a choice of Kan complex Y. See Exercise 4.10.

Kan fibrations are determined via lifting conditions, and so compose and are stable under pullbacks.

Lemma 4.9. *The composition of two Kan fibrations is a Kan fibration.*

Proof. Exercise 4.11 □

Lemma 4.10. *Let the following diagram be a pullback square of simplicial sets*

$$\begin{tikzcd} W \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, two heads, "p"] \\ Z \arrow[r] & X \end{tikzcd} .$$

If p is a Kan fibration, then q is also a Kan fibration.

Proof. Exercise 4.12 □

The pullback property also has some additional useful implications about Kan fibrations.

Lemma 4.11. *Let $p \colon Y \to X$ be a Kan fibration. Assume $Y \xrightarrow{q} A \xrightarrow{i} X$ is a factorization of p, such that i is an inclusion of simplicial sets. Then q is a Kan fibration.*

Proof. The inclusion assumption implies that the following diagram is a pullback square

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, two heads, "p"] \\ A \arrow[r, hook, "i"] & X \end{tikzcd} ,$$

and so the result follows from Lemma 4.10 and that p is a Kan fibration. □

We now have the following powerful technical result about pullbacks and Kan fibrations, analogous to Lemma 4.4. Unfortunately, a proof requires developing some theory and goes beyond our aims. See [Hir03, Proposition 13.3.9] for a proof of the case $n = 2$, which implies the general case by induction.

Lemma 4.12. *Let $n \geq 2$. Assume we have the following diagram of Kan complexes,*

$$\begin{tikzcd} A_1 \arrow[r, two heads] \arrow[d, "f_1"', "\simeq"] & A_{1,2} \arrow[d, "f_{1,2}"', "\simeq"] & \dots \arrow[l, two heads] \arrow[r, two heads] & A_{n-1,n} \arrow[d, "f_{n-1,n}"', "\simeq"] & A_n \arrow[l, two heads] \arrow[d, "f_n"', "\simeq"] \\ B_1 \arrow[r, two heads] & B_{1,2} & \dots \arrow[l, two heads] \arrow[r, two heads] & B_{n-1,n} & B_n \arrow[l, two heads] \end{tikzcd},$$

where the horizontal maps are Kan fibrations and the vertical maps are homotopy equivalences. Then the induced map on limits is also a homotopy equivalence.

Remark 4.13. In fact [Hir03, Proposition 13.3.9] states that if $n = 2$ in Lemma 4.12, then it suffices for $A_1 \to A_{1,2}$ and $B_1 \to B_{1,2}$ to be Kan fibrations for the result to hold.

Lemma 4.14 ([Lur26, Proposition 00XC])**.** *Assume we have the following commutative diagram of Kan complexes*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Z \arrow[dl, two heads, "q"] \\ & X & \end{tikzcd},$$

where the maps p and q are Kan fibrations. Then f is a homotopy equivalence if and only if for every x in X_0 the induced map on pre-images (which we also call fibers) $f_x \colon p^{-1}(x) \to q^{-1}(x)$ is a homotopy equivalence.

We now study how mapping spaces (Definition 1.28) interact with Kan fibrations. This is in fact one of the most important and powerful results about Kan fibrations. Unfortunately, the proof is very technical, so we refer the reader to [GJ09, Corollary 5.3] as a suitable source.

Proposition 4.15. *Let X be a Kan complex, and $i \colon A \hookrightarrow B$ be an inclusion of simplicial sets (a subsimplicial set). Then the induced map of simplicial sets*

$$i^* \colon \mathrm{Map}(B, X) \twoheadrightarrow \mathrm{Map}(A, X)$$

is a Kan fibration.

There are particularly relevant examples of this result.

Lemma 4.16. *Let X be a Kan complex, and A be a simplicial set. Then* $\mathrm{Map}(A, X)$ *is a Kan complex.*

Proof. Exercise 4.15 □

We can use this example to get some new equivalences of Kan complexes.

Lemma 4.17. *Let K be a Kan complex. Then for every $n \geq 0$, the maps $\mathbb{D}^0 \xrightarrow{0} \mathbb{D}^n \to \mathbb{D}^0$ induce homotopy equivalences of Kan complexes* $\mathrm{Map}(\mathbb{D}^0, K) \simeq \mathrm{Map}(\mathbb{D}^n, K)$*.*

Proof. Exercise 4.17 □

For the next example, we introduce new notions.

Definition 4.18 (Path space)**.** Let K be a Kan complex and let x, y be two points in K, corresponding to two maps $x, y \colon \mathbb{D}^0 \to K$. We define the *path space* between x and y as the following pullback

$$\begin{tikzcd} \mathcal{P}\mathrm{ath}_K(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(\mathbb{D}^1, K) \arrow[d, two heads, "(0^*,1^*)"] \\ \mathbb{D}^0 \arrow[r, "(x,y)"'] & \mathrm{Map}(\partial\mathbb{D}^1, K) \end{tikzcd},$$

where for the bottom map we use the explicit characterization of $\mathrm{Map}(\partial\mathbb{D}^1, K)_0 \cong K_0 \times K_0$, relying on the computation $\partial\mathbb{D}^1 \cong \mathbb{D}^0 \coprod \mathbb{D}^0$ (Lemma 1.47) and Proposition 1.50.

Lemma 4.19. *Let K be a Kan complex and let x, y be two points in K. Then the path space $\mathcal{P}\mathrm{ath}_K(x, y)$ is a Kan complex.*

Proof. Exercise 4.18 □

We end this section with a large class of examples of Kan fibrations, which will be useful in the next section. This will require analyzing path-components of simplicial sets, which we now define.

Definition 4.20. Let $\pi_0 \colon s\mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ be the left adjoint to the inclusion $\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S}\mathrm{et}$, which regards sets as constant simplicial sets.

Remark 4.21. For a simplicial set S, $\pi_0(S)$ is the set of *path-components* of S and is explicitly described in Definition 2.40 as the coequalizer of the two face maps $d_0, d_1 \colon S_1 \rightrightarrows S_0$.

Definition 4.22 (Inclusion of path-components)**.** A map $i \colon A \to B$ of Kan complexes is an *inclusion of path-components* if $\pi_0(i) \colon \pi_0(A) \to \pi_0(B)$ is an injection and the following diagram is a pullback

$$\begin{tikzcd} A \arrow[r, "\pi_0"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\ulcorner", very near start] & \pi_0(A) \arrow[d, hook, "\pi_0(i)"] \\ B \arrow[r, "\pi_0"] & \pi_0(B) \end{tikzcd} .$$

Intuition 4.23. Intuitively, we can write the simplicial set B as a disjoint union of its path-components, meaning there is an isomorphism of simplicial sets $B \cong \coprod_{[b] \in \pi_0(B)} B_b$, where B_b is the sub-simplicial set of B generated by all vertices (in the sense of Example 1.32) b' satisfying $[b] = [b']$.

From this perspective we can very intuitively say that an inclusion of path-components is simply a choice of subset $A' \subseteq \pi_0(B)$ and $A = \coprod_{b \in A'} B_b$, at which point $i \colon A \to B$ is simply $\coprod_{b \in A'} B_b \hookrightarrow \coprod_{b \in \pi_0(B)} B_b$, meaning it is indeed an inclusion of path-components.

We now have the following lemmas and results.

Lemma 4.24. *Let $n \geq 1, 0 \leq l \leq n$. Then $\pi_0(\mathbb{D}^n) \cong \pi_0(\mathbb{L}^n_l) \cong \{[0]\}$.*

Proof. Any two vertices in $\mathbb{L}^n_l$ are connected by a chain of 1-simplices. Indeed in the cases $n \geq 3$, any two vertices are directly connected, and in the case of $n = 2$, there is always a chain of length 2 (Example 1.39). We have a similar argument for $\mathbb{D}^n$ (Exercise 4.19). Hence, $\pi_0(\mathbb{D}^n) \cong \pi_0(\mathbb{L}^n_l) \cong \{[0]\}$. □

We now have the following result about maps into constant simplicial sets, as defined in Definition 1.14.

Lemma 4.25. *Let S be a constant simplicial set. Then for $n \geq 1, 0 \leq l \leq n$, every map $\mathbb{D}^n \to S$ and $\mathbb{L}^n_l \to S$ is constant.*

Proof. We prove the case for $\mathbb{L}^n_l$ and leave the case for $\mathbb{D}^n$ as an exercise (Exercise 4.20). A map $f \colon \mathbb{L}^n_l \to S$ induces a map on path-components $\pi_0(\mathbb{L}^n_l) \to \pi_0(S)$ (Definition 4.20). As S is constant, $\pi_0(S) = S_0$. Hence, by Lemma 4.24, $\pi_0(f) \colon \pi_0(\mathbb{L}^n_l) \to S_0$ is necessarily constant with image s in S_0.

Now, let σ in $(\mathbb{L}^n_l)_k$. Then, by the previous paragraph $f_0(0^*\sigma) = s$. Now, by naturality $0^* f_k(\sigma) = f_0(0^*\sigma) = s$, but $0^* \colon S_k \to S_0$ is a bijection, so $f_k(\sigma) = s$ and we are done. □

Lemma 4.26. *Let S, T be constant simplicial sets. Then any map $f \colon S \to T$ is a Kan fibration.*

Proof. We need to show that for every $n \geq 1, 0 \leq k \leq n$, the following diagram admits a lift

$$\begin{tikzcd} \mathbb{L}^n_k \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ \mathbb{D}^n \arrow[r] & T \end{tikzcd} .$$

By Lemma 4.25, this diagram will factor as follows

$$\begin{tikzcd} \mathbb{L}^n_k \arrow[r] \arrow[d] & \mathbb{D}^0 \arrow[r, "s"] \arrow[d, equal] & S \arrow[d, "f"] \\ \mathbb{D}^n \arrow[r] & \mathbb{D}^0 \arrow[r, "t"] & T \end{tikzcd} .$$

Now, the desired lift is given by the map $\mathbb{D}^n \to \mathbb{D}^0 \xrightarrow{s} S$, and so we are done. □

Intuition 4.27. A Kan fibration is seen as a "dependent space", i.e. a space that varies over another space. So, if the spatial structure is trivial, then intuitively one would expect the Kan fibration condition to follow automatically. This is the content of Lemma 4.26.

Proposition 4.28. *Let $i\colon A \hookrightarrow B$ be an inclusion of path-components. Then i is a Kan fibration.*

Proof. By Lemma 4.26, the map $\pi_0(i)\colon \pi_0(A) \hookrightarrow \pi_0(B)$ is a Kan fibration. By Definition 4.22, i is a pullback of $\pi_0(i)$. So, by Lemma 4.10, the map $i\colon A \to B$ is a Kan fibration. □

Intuition 4.29. Following Intuition 4.23, an inclusion of path-components $A \to B$ can be seen as a map of the form $\coprod_{i\in I_0} B_i \to \coprod_{i \in I} B_i$. As $\mathbb{L}^n_k, \mathbb{D}^n$ are connected, for any commutative diagram of the form

$$\begin{tikzcd} \mathbb{L}^n_k \arrow[r] \arrow[d, hook] & A \arrow[d, hook, "i"] \\ \mathbb{D}^n \arrow[r] & B \end{tikzcd} ,$$

the map $\mathbb{L}^n_k \to A$ needs to factor through a specific path-component B_i of A. Moreover, by commutativity, the image of $\mathbb{D}^n \to B$ also needs to factor through B_i. Corestricting the right-hand side to B_i, we hence only need to find a lift for the $\mathrm{id}_{B_i}\colon B_i \to B_i$, which of course always exists.

Exercises.

Exercise 4.1. Let $p\colon Y \to X$ be a map of simplicial sets and let $x \in X_0$, regarded as a map

$$x\colon \mathbb{D}^0 \to X.$$

Define the fiber, or pre-image, of p over x as a pullback

$$\begin{tikzcd} p^{-1}(x) \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \mathbb{D}^0 \arrow[r, "x"'] & X. \end{tikzcd}$$

Describe explicitly the n-simplices of $p^{-1}(x)$.

Exercise 4.2. Let K be a Kan complex and let $x, y\colon \mathbb{D}^0 \to K$ be two points. Using Definition 4.18, describe the set $\mathcal{P}\mathrm{ath}_K(x, y)_0$ explicitly.

Exercise 4.3. Let S be a simplicial set. Using the description of $\pi_0(S)$ as the coequalizer of

$$d_0, d_1\colon S_1 \rightrightarrows S_0,$$

explain why a map from S to a constant simplicial set T is the same as a function

$$\pi_0(S) \to T_0.$$

Exercise 4.4. Suppose we have a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X. & \end{tikzcd}$$

For $x \in X_0$, describe the induced map on fibers

$$f_x \colon p^{-1}(x) \to q^{-1}(x)$$

as a map between pullbacks.

Exercise 4.5. Prove Lemma 4.1. That is, assume we have a commutative diagram of sets

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X. & \end{tikzcd}$$

Prove that f is a bijection if and only if, for every $x \in X$, the induced map

$$f_x \colon p^{-1}(x) \to q^{-1}(x)$$

is a bijection.

Exercise 4.6. Fill in the details of Lemma 4.2. Namely, verify that the functor

$$H \colon [1] \times I(1) \to I(1)$$

used in the proof really defines a homotopy

$$\mathbb{D}^1 \times N(I(1)) \to N(I(1))$$

from $N(0 \circ t)$ to $\mathrm{id}_{N(I(1))}$.

Exercise 4.7. Work out Example 4.3 explicitly. Compute the fibers of

$$N(0) \colon \mathbb{D}^0 \to N(I(1))$$

over the two vertices $0, 1 \in N(I(1))_0$. Explain why the fiber over 1 gives a map

$$\emptyset \to \mathbb{D}^0$$

which is not a homotopy equivalence.

Exercise 4.8. Prove Lemma 4.4 in the case $n = 2$. In other words, assume we have a commutative diagram of sets

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\cong"'] & C \arrow[d, "\cong"] & B \arrow[l] \arrow[d, "\cong"] \\ A' \arrow[r] & C' & B'. \arrow[l] \end{tikzcd}$$

Prove that the induced map

$$A \times_C B \to A' \times_{C'} B'$$

is a bijection.

Exercise 4.9. Work out Example 4.5 explicitly. Show that

$$\mathbb{D}^0 \times^{0,1}_{N(I(1))} \mathbb{D}^0$$

is the empty simplicial set, while the corresponding lower pullback is $\mathbb{D}^0$.

Exercise 4.10. Complete the argument in Remark 4.8. Namely, show directly from the definitions that a map

$$Y \to \mathbb{D}^0$$

is a Kan fibration if and only if Y is a Kan complex.

Exercise 4.11. Prove Lemma 4.9: the composition of two Kan fibrations is a Kan fibration.

Exercise 4.12. Prove Lemma 4.10: a pullback of a Kan fibration is again a Kan fibration.

Exercise 4.13. Let
$$p\colon Y \twoheadrightarrow X$$
be a Kan fibration and let $x \in X_0$. Prove that the fiber
$$p^{-1}(x)$$
is a Kan complex.

Exercise 4.14. Prove that if K and L are Kan complexes, then their product
$$K \times L$$
is again a Kan complex.

Exercise 4.15. Use Proposition 4.15 to prove Lemma 4.16: if K is a Kan complex and A is any simplicial set, then
$$\mathrm{Map}(A, K)$$
is a Kan complex.

Exercise 4.16. Use Proposition 4.15 to show that, for every Kan complex K, the endpoint map
$$(0^*, 1^*)\colon \mathrm{Map}(\mathbb{D}^1, K) \to \mathrm{Map}(\partial\mathbb{D}^1, K)$$
is a Kan fibration.

Exercise 4.17. Prove Lemma 4.17, that is, for $n \geq 0$ and K a Kan complex, the maps $\mathbb{D}^0 \xrightarrow{0} \mathbb{D}^n \to \mathbb{D}^0$ induce homotopy equivalences of Kan complexes $\mathrm{Map}(\mathbb{D}^0, K) \simeq \mathrm{Map}(\mathbb{D}^n, K)$.

Exercise 4.18. Prove Lemma 4.19: for every Kan complex K and every pair of points $x, y \in K_0$, the path space
$$\mathcal{P}\mathrm{ath}_K(x, y)$$
is a Kan complex.

Exercise 4.19. Complete Lemma 4.24: for $n \geq 1$, one has
$$\pi_0(\mathbb{D}^n) \cong \{[0]\}.$$

Exercise 4.20. Use the Yoneda lemma to prove the missing part of Lemma 4.25: if S is a constant simplicial set, then every map
$$\mathbb{D}^n \to S$$
is constant for $n \geq 1$.

Exercise 4.21. Let
$$p\colon E \twoheadrightarrow B$$
be a Kan fibration of Kan complexes, and let
$$f\colon A \to B$$
be a homotopy equivalence of Kan complexes. Using the case $n = 2$ of Lemma 4.12 and Remark 4.13, prove that the induced map
$$A \times_B E \to E$$
is a homotopy equivalence.

Exercise 4.22. Compute $N(I(1))_0$, $N(I(1))_1$, and $N(I(1))_2$ explicitly. Identify the non-degenerate simplices in dimensions 0, 1, and 2.

Exercise 4.23. Compute
$$\pi_0(N(I(1))), \qquad \pi_0(\mathbb{D}^0 \coprod \mathbb{D}^0), \qquad \pi_0(\partial\mathbb{D}^1), \qquad \pi_0(\partial\mathbb{D}^2).$$

Exercise 4.24. Compute the fibers of the map

$$N(0)\colon \mathbb{D}^0 \to N(I(1))$$

over the vertices 0 and 1 of $N(I(1))$.

Exercise 4.25. Show that the map

$$N(0)\colon \mathbb{D}^0 \to N(I(1))$$

is not a Kan fibration. Construct an explicit lifting problem involving

$$\mathbb{L}^1_0 \hookrightarrow \mathbb{D}^1$$

which admits no lift.

Exercise 4.26. Let S be a constant simplicial set. Compute

$$\mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\mathbb{D}^n, S) \qquad \text{and} \qquad \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\mathbb{L}^n_l, S)$$

for $n \geq 1$ and $0 \leq l \leq n$.

Exercise 4.27. Let S be a constant simplicial set and let $x, y \in S_0$. Compute

$$\mathcal{P}\mathrm{ath}_S(x,y).$$

Show that it is isomorphic to $\mathbb{D}^0$ if $x = y$, and is the empty simplicial set if $x \neq y$.

Exercise 4.28. Compute the vertices of the path spaces

$$\mathcal{P}\mathrm{ath}_{N(I(1))}(0,0), \qquad \mathcal{P}\mathrm{ath}_{N(I(1))}(0,1), \qquad \mathcal{P}\mathrm{ath}_{N(I(1))}(1,0), \qquad \mathcal{P}\mathrm{ath}_{N(I(1))}(1,1).$$

Exercise 4.29. Let $B = \mathbb{D}^0 \coprod \mathbb{D}^0$. List all inclusions of path-components

$$A \hookrightarrow B.$$

For each one, compute $\pi_0(A) \to \pi_0(B)$.

5. Reedy Fibrancy

As explained in Section 3, our simplicial spaces should behave like "spaces" (i.e. Kan complexes) in what we already called the spatial (or "vertical") direction. Naively, we might hence expect to impose a level-wise Kan complex condition. However, later on we aim to impose additional conditions. For example, as we showed in Proposition 2.13, the nerve $N\mathcal{C}$ of a category $\mathcal{C}$ satisfies the Segal condition, which, as we saw in Proposition 2.8, is in particular a statement about the pullback $N\mathcal{C}_1 \times_{N\mathcal{C}_0} N\mathcal{C}_1$. We want an analogous statement in the context of simplicial spaces, meaning we want to analyze pullbacks of Kan complexes $X_1 \times_{X_0} X_1$. However, Kan complexes operate in the realm of homotopy theory and so all statements about them, including pullbacks, should be invariant under homotopy equivalences. Here we can benefit from the insight of Section 4.

As we saw in Lemma 4.12, if the maps $0^*, 1^*\colon X_1 \to X_0$ are Kan fibrations, then the pullback behaves correctly. This is one of the situations where the various spaces constructing a simplicial space need to be dependent (in the sense of Section 4). For that reason we now introduce the Reedy fibrancy condition.

Definition 5.1 (Reedy fibrancy)**.** A map of simplicial spaces $p\colon Y \to X$ is a *Reedy fibration* if for every $n \geq 0, k \geq 1, 0 \leq i \leq k$, the following diagram lifts

$$\begin{tikzcd}
\partial\Delta^n \times \mathbb{D}^k \coprod_{\partial\Delta^n \times \mathbb{L}^k_i} \Delta^n \times \mathbb{L}^k_i \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
\Delta^n \times \mathbb{D}^k \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

A simplicial space X is *Reedy fibrant* if the map $X \to \Delta^0$ is a Reedy fibration.

The Reedy fibrancy condition has many powerful implications and equivalent characterizations. However, proving that in full generality requires some more advanced machinery. We refer the reader to [Rez01, Section 2.5] for a review and to [Hir03, Proposition 9.3.1, Theorem 15.3.4] for more detailed proofs.

Lemma 5.2. *Let X be a simplicial space. Then the following are equivalent:*

(1) X is Reedy fibrant.
(2) For every $n \geq 0$, the induced map of mapping spaces
$$\mathrm{Map}(\Delta^n, X) \twoheadrightarrow \mathrm{Map}(\partial\Delta^n, X)$$
is a Kan fibration.
(3) For every inclusion of simplicial spaces $A \hookrightarrow B$, the induced map of mapping spaces
$$\mathrm{Map}(B, X) \twoheadrightarrow \mathrm{Map}(A, X)$$
is a Kan fibration.
(4) For every inclusion of simplicial spaces $A \hookrightarrow B$, the induced map of mapping spaces
$$[\![B, X]\!] \twoheadrightarrow [\![A, X]\!]$$
is a Reedy fibration.

We can use the Reedy fibrancy condition to get several important dependent Kan complexes, in the sense of Section 4. Let us see various manifestations of this dependency condition.

Example 5.3. If X is a Reedy fibrant simplicial space, then X_n is a Kan complex. See Exercise 5.6.

Example 5.4. Let X be a Reedy fibrant simplicial space. Then the maps $(0^*, 1^*)\colon X_1 \to X_0 \times X_0$ and $0^*, 1^*\colon X_1 \to X_0$ are Kan fibrations. See Exercise 5.7.

Example 5.5. Let X be a Reedy fibrant simplicial space and $n \geq 2$. Then the map $(0^*, ..., n^*)\colon X_n \to (X_0)^{n+1}$ is a Kan fibration. See Exercise 5.8.

These observations show how Reedy fibrancy helps realize our goal of having a collection of Kan complexes with a well-behaved notion of pullback. Let us end with some examples.

Lemma 5.6. *Let A be a simplicial set and $n \geq 0$. Then $\mathrm{Map}(\Delta^n, \mathrm{disc}(A))$ is a constant simplicial set on the set $\mathrm{Hom}(\Delta^n, \mathrm{disc}(A))$.*

Proof. Exercise 5.9. □

Next, we have the following lemma, which is primarily a combinatorial exercise.

Lemma 5.7. *Let A be a simplicial set and $n \geq 0$. Then $\mathrm{Map}(\partial\Delta^n, \mathrm{disc}(A))$ is a constant simplicial set on the set $\mathrm{Hom}(\partial\Delta^n, \mathrm{disc}(A))$.*

Proof. We omit the general combinatorial proof. For the cases $n = 1$ and $n = 2$, see Exercises 5.10 and 5.11. □

Proposition 5.8. *Let A be a simplicial set. Then $\mathrm{disc}(A)$ is Reedy fibrant.*

Proof. By Lemma 5.2, it suffices to show that for every $n \geq 0$, the map
$$\mathrm{Map}(\Delta^n, \mathrm{disc}(A)) \twoheadrightarrow \mathrm{Map}(\partial\Delta^n, \mathrm{disc}(A))$$
is a Kan fibration. By Lemmas 5.6 and 5.7 both domain and codomain are constant simplicial sets, and so the desired result follows from Lemma 4.26. □

Example 5.9. Combining Notation 3.10 and Proposition 5.8 it follows that the simplicial spaces $\Delta^n, \partial\Delta^n$ are Reedy fibrant for every $n \geq 0$, and Sp^n is Reedy fibrant for $n \geq 2$.

Example 5.10. The simplicial space $\mathbb{D}^1$, which is our notation for $\mathrm{const}(\mathbb{D}^1)$, is not Reedy fibrant. Indeed, following Example 5.3, the simplicial set $\mathrm{const}(\mathbb{D}^1)_0 = \mathbb{D}^1 = N([1])$ would need to be a Kan complex, which, following Lemma 3.1, would require $[1]$ to be a groupoid, which is not the case.

Example 5.11. Recall from Lemma 3.1 that $N(I(1))$ is in fact a Kan complex. Yet, the simplicial space $N(I(1))$, which is our notation for $\mathrm{const}(N(I(1)))$, is not Reedy fibrant. Indeed, following Example 5.4, the diagonal map $\Delta\colon N(I(1)) \to N(I(1)) \times N(I(1))$ needs to be a Kan fibration. However, recalling that $\mathbb{L}^1_0 \cong \mathbb{D}^0$, by Example 1.38, the diagram

$$\begin{tikzcd} \mathbb{L}^1_0 \arrow[r, "0"] \arrow[d] & N(I(1)) \arrow[d, "\Delta"] \\ \mathbb{D}^1 \arrow[r, "{(01,00)}"] & N(I(1)) \times N(I(1)) \end{tikzcd}$$

does not admit a lift (Exercise 5.17).

Exercises.

Exercise 5.1. Let $p\colon Y \to X$ be a map of simplicial spaces. Unpack the Reedy fibration lifting condition in the special case $n = 0$. Show that the condition says precisely that the map of simplicial sets

$$Y_0 \to X_0$$

is a Kan fibration.

Exercise 5.2. Let $p\colon Y \to X$ be a map of simplicial spaces. Unpack the Reedy fibration lifting condition in the special case $n = 1$. In particular, identify the map

$$\mathrm{Map}(\Delta^1, Y) \to \mathrm{Map}(\partial\Delta^1, Y) \times_{\mathrm{Map}(\partial\Delta^1, X)} \mathrm{Map}(\Delta^1, X)$$

and explain how it encodes compatibility with the two endpoint maps.

Exercise 5.3. Let X be a simplicial space. Use the identification

$$\partial\Delta^1 \cong \Delta^0 \coprod \Delta^0$$

to identify the map

$$\mathrm{Map}(\Delta^1, X) \to \mathrm{Map}(\partial\Delta^1, X)$$

with the map

$$(0^*, 1^*)\colon X_1 \to X_0 \times X_0.$$

Exercise 5.4. Let A be a simplicial set. Compare the simplicial spaces

$$\mathrm{disc}(A) \qquad \text{and} \qquad \mathrm{const}(A)$$

from the perspective of Reedy fibrancy. In particular, compute their n-th vertical simplicial sets and explain why Reedy fibrancy tests different things for these two inclusions.

Exercise 5.5. Let X be a Reedy fibrant simplicial space. Explain why it is not enough to say that each X_n is a Kan complex. Which additional condition involving

$$X_1 \to X_0 \times X_0$$

is forced by Reedy fibrancy?

Exercise 5.6. Use the inclusion $\emptyset \hookrightarrow \Delta^n$ to prove Example 5.3: if X is a Reedy fibrant simplicial space, then X_n is a Kan complex for every $n \geq 0$.

Exercise 5.7. Use the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$ to prove Example 5.4: if X is a Reedy fibrant simplicial space, then the map

$$(0^*, 1^*)\colon X_1 \to X_0 \times X_0$$

is a Kan fibration. Deduce that both maps

$$0^*, 1^*\colon X_1 \to X_0$$

are Kan fibrations.

Exercise 5.8. Use the inclusion $0 + ... + n\colon \Delta^0 \coprod ... \coprod \Delta^0 \hookrightarrow \Delta^n$ to prove Example 5.5: if X is a Reedy fibrant simplicial space, then for every $n \geq 2$ the map

$$X_n \to (X_0)^{n+1}$$

induced by the vertices of Δ^n is a Kan fibration.

Exercise 5.9. Prove Lemma 5.6: for every $n \geq 0$ and simplicial set A, the simplicial set

$$\mathrm{Map}(\Delta^n, \mathrm{disc}(A))$$

is constant on the set

$$\mathrm{Hom}(\Delta^n, \mathrm{disc}(A)).$$

Exercise 5.10. Prove Lemma 5.7 in the case $n = 1$. That is, show that for every simplicial set A,

$$\mathrm{Map}(\partial\Delta^1, \mathrm{disc}(A))$$

is a constant simplicial set.

Exercise 5.11. Prove Lemma 5.7 in the case $n = 2$. Concretely, show that

$$\mathrm{Map}(\partial\Delta^2, \mathrm{disc}(A))$$

is constant by describing its l-simplices and checking that they do not depend on l.

Exercise 5.12. Show that the simplicial spaces

$$\Delta^n, \qquad \partial\Delta^n, \qquad \mathrm{Sp}^n$$

are Reedy fibrant for every $n \geq 0$ (or $n \geq 2$ in the case Sp^n).

Exercise 5.13. Let S be a Kan complex. Show that $\mathrm{const}(S)$ need not be Reedy fibrant.

Exercise 5.14. Let X be a simplicial space. Compute

$$\mathrm{Map}(\partial\Delta^1, X)_k$$

for arbitrary $k \geq 0$, and identify it with

$$(X_0 \times X_0)_k.$$

Exercise 5.15. Compute

$$\mathrm{Map}(\Delta^2, \Delta^1).$$

Is this simplicial set constant?

Exercise 5.16. Let S be a simplicial set. Compute

$$\mathrm{Map}(\Delta^n, \mathrm{const}(S))$$

as a simplicial set. Then compute the map

$$\mathrm{Map}(\Delta^1, \mathrm{const}(S)) \to \mathrm{Map}(\partial\Delta^1, \mathrm{const}(S)).$$

Show that this map is the diagonal

$$S \to S \times S.$$

Exercise 5.17. Work out the lifting obstruction in Example 5.11. Namely, show that the diagram

$$\begin{tikzcd}
\mathbb{L}^1_0 \arrow[r, "0"] \arrow[d] & N(I(1)) \arrow[d, "\Delta"] \\
\mathbb{D}^1 \arrow[r, "(01{,}00)"'] & N(I(1)) \times N(I(1))
\end{tikzcd}$$

admits no lift.

Exercise 5.18. Let $X = \mathrm{const}(\mathbb{D}^1)$. Compute X_0 and explain explicitly why X_0 is not a Kan complex.

Exercise 5.19. Let $X = \mathrm{disc}(\mathbb{D}^1)$. Compute X_0 and X_1 as simplicial sets. Explain why both are Kan complexes, even though $\mathbb{D}^1$ itself is not a Kan complex.

Exercise 5.20. Let X be a Reedy fibrant simplicial space. Compute the fiber of

$$(0^*, 1^*) \colon X_1 \to X_0 \times X_0$$

over a pair of vertices $(x, y) \in (X_0)_0 \times (X_0)_0$. Explain why this fiber is a Kan complex.

6. Interlude: Uniqueness vs. Contractibility

Many properties in category theory (composition, inverses, ...) involve uniqueness. Hence, before we proceed with our study of higher categories it is instructive to review the concept of "uniqueness" in a homotopical context.

Recall that in set theory, for a given set X we can define formulas $\varphi(x)$ on X. An easy example is the formula $n > 3$ on the set $\mathbb{N}$. Given such a formula φ on X, we can construct the subset of X consisting of all elements that satisfy φ, i.e. the subset $\{x \in X \mid \varphi(x)\}$. In our example this would simply be $\{n \in \mathbb{N} \mid n > 3\} = \{4, 5, ...\}$. We now have the following classical definition.

Definition 6.1. Let X be a set and let φ be a property on X. We say $x_0 \in X$ satisfies φ uniquely if x_0 is the only element in $\{x \in X \mid \varphi(x)\}$.

These definitions are used all throughout mathematics and category theory.

Example 6.2. Let $\mathcal{C}$ be a category and let $f \colon X \to Y$, $g \colon Y \to Z$ be two morphisms. Then the set $\{h \colon X \to Z \mid h \text{ is the composition of } f \text{ and } g\}$ has a unique element, namely the composite $g \circ f$.

In the context of homotopy theory, these notions of uniqueness need to be adapted. Indeed, we are replacing sets with Kan complexes and bijections with homotopy equivalences. We hence need a homotopical analogue of uniqueness.

Definition 6.3 (Contractibility)**.** A Kan complex K is called *contractible* if the unique map $K \to \mathbb{D}^0$ (Lemma 1.20) is a homotopy equivalence.

Remark 6.4. Contractible Kan complexes are precisely the homotopical analogue of singleton sets.

Previously we considered subsets associated to a given formula φ. This can be directly generalized to the homotopical setting.

Definition 6.5 (Sub-Kan complex)**.** Let K be a Kan complex. A *sub-Kan complex* $L \subseteq K$ is a sub-simplicial set that is also a Kan complex.

Definition 6.6 (Homotopically unique)**.** Let K be a Kan complex. A *homotopical property* on K is a sub-Kan complex $L \subseteq K$. It is *homotopically unique* if L is contractible.

Homotopical uniqueness and contractibility are sufficient when we have a property on a single Kan complex and we want to analyze whether the property is homotopically unique. However, as we discussed in Section 4, we are often in a dependent situation, where we have a family of objects or properties that vary over some base. In the context of sets, for example, we have the following results.

Lemma 6.7. *Let $f \colon Y \to Z$ be a map of sets. Then f is a bijection if and only if for every $z \in Z$ the pre-image $f^{-1}(z)$ is a singleton.*

Proof. Exercise 6.7 $\square$

Intuition 6.8. We can think of the set $Y = \coprod_{z \in Z} f^{-1}(z)$ as a disjoint union of the pre-images. From this perspective, we can think of bijections as a "global form of uniqueness over Z" and Lemma 6.7 as the statement that this global form of uniqueness is equivalent to a "local form of uniqueness" for every z in Z.

We want an analogous result for Kan complexes, where bijections are replaced by homotopy equivalences and singletons are replaced by contractible Kan complexes. However, the following example shows that this will fail without some assumption.

Example 6.9. Let $0\colon \mathbb{D}^0 \to N(I(1))$ be the homotopy equivalence defined in Example 4.3. Then the pre-image of 1 is the empty set, which is not contractible. Hence, being a global equivalence does not imply local homotopical uniqueness.

Building on Section 4, we want to characterize dependent homotopical uniqueness as a suitable condition on Kan fibrations, as they model dependent structures.

Definition 6.10 (Trivial fibration)**.** A map of simplicial sets $f\colon Y \to Z$ is a *trivial fibration* if it is a Kan fibration and a homotopy equivalence.

Remark 6.11. This definition is indeed a simultaneous generalization of contractibility and dependence. In particular, a trivial Kan fibration $K \to \mathbb{D}^0$ is precisely the data of a contractible Kan complex K (Exercise 6.14).

We now have the following very powerful result about trivial fibrations, demonstrating its suitability for dependent homotopical uniqueness. For the proof, we refer the reader to [GJ09, Theorem I.11.2].

Proposition 6.12. *Let $f\colon Y \to X$ be a map of simplicial sets. Then the following are equivalent.*
- *f is a trivial fibration.*
- *f is a Kan fibration and for every x in X_0, the pre-image $f^{-1}(x)$ is a contractible Kan complex.*
- *For every $n \geq 0$ and commutative diagram*

$$\begin{tikzcd} \partial\mathbb{D}^n \arrow[r] \arrow[d] & Y \arrow[d, two heads, "f", "\simeq"'] \\ \mathbb{D}^n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

  there exists a lift.

Before we apply dependent homotopical uniqueness in the next section, let us record results analogous to Lemmas 4.9 and 4.10.

Lemma 6.13. *Let the following diagram be a pullback square of simplicial sets*

$$\begin{tikzcd} W \arrow[r] \arrow[d, two heads, "q"', "\simeq"] & Y \arrow[d, two heads, "p", "\simeq"'] \\ Z \arrow[r] & X \end{tikzcd}.$$

If p is a trivial Kan fibration, then q is also a trivial Kan fibration.

Proof. Exercise 6.15 □

Lemma 6.14. *Let $f\colon Y \to Z$ and $g\colon Z \to X$ be maps of simplicial sets. If f and g are trivial Kan fibrations, then so is the composition $g \circ f\colon Y \to X$.*

Proof. Exercise 6.16 □

Finally, let us note that these results also generalize to simplicial spaces.

Definition 6.15. A map of simplicial spaces $f\colon Y \to X$ is a *homotopy equivalence* if $f_n\colon Y_n \to X_n$ is a homotopy equivalence for every $n \geq 0$. A map of simplicial spaces $f\colon Y \to X$ is a *trivial Reedy fibration* if it is a Reedy fibration and a homotopy equivalence.

Remark 6.16. Categories are usually defined in a set-theoretic context, and the notion of uniqueness in that foundation matches the intended categorical applications. As we saw in this section, homotopical uniqueness is more intricate in the set-theoretic context, and requires the contractibility of Kan complexes. This raises the question whether it is possible to adjust the underlying foundation so that uniqueness in that foundation matches homotopical uniqueness. This has been successfully achieved via the new foundation of *homotopy type theory* [Uni13].

Exercises.

Exercise 6.1. Let K be a Kan complex. Unpack what it means for K to be contractible in terms of maps

$$K \to \mathbb{D}^0 \qquad \text{and} \qquad \mathbb{D}^0 \to K$$

and homotopies between their composites and the relevant identity maps.

Exercise 6.2. Let S be a subset of K_0, where K is a Kan complex. Consider the sub-simplicial set $\bar{S} \subseteq K$ generated by S. Give a criterion for $\bar{S}$ to define a homotopical property on K, and then state what it would mean for this property to be homotopically unique.

Exercise 6.3. Let

$$f\colon Y \to X$$

be a map of simplicial sets and let $x \in X_0$. Write the fiber $f^{-1}(x)$ as a pullback. Then state, using Proposition 6.12, what it means for the fiber over x to satisfy local homotopical uniqueness.

Exercise 6.4. Unpack the definition of a trivial fibration. Then explain why a trivial fibration

$$K \to \mathbb{D}^0$$

is the same thing as a contractible Kan complex.

Exercise 6.5. In the lifting characterization of trivial fibrations from Proposition 6.12, compute the cases $n = 0$ and $n = 1$ explicitly. In particular, use

$$\partial\mathbb{D}^0 = \emptyset \qquad \text{and} \qquad \partial\mathbb{D}^1 \cong \mathbb{D}^0 \coprod \mathbb{D}^0.$$

What do these two lifting conditions say about vertices and edges?

Exercise 6.6. Let

$$f\colon Y \to X$$

be a map of simplicial spaces. Unpack the definition of a homotopy equivalence of simplicial spaces. Then unpack the definition of a trivial Reedy fibration.

Exercise 6.7. Prove Lemma 6.7: if $f\colon Y \to Z$ is a map of sets, then f is a bijection if and only if, for every $z \in Z$, the pre-image

$$f^{-1}(z)$$

is a singleton.

Exercise 6.8. Let $f\colon Y \to Z$ be a map of sets. Write Y as a disjoint union of its fibers:

$$Y \cong \coprod_{z \in Z} f^{-1}(z).$$

Use this description to give another proof of Lemma 6.7.

Exercise 6.9. Show that the empty simplicial set is not contractible. That is, show that the unique map

$$\emptyset \to \mathbb{D}^0$$

is not a homotopy equivalence.

Exercise 6.10. Work out the example following Lemma 6.7. Let

$$0\colon \mathbb{D}^0 \to N(I(1))$$

be the homotopy equivalence from Example 4.3. Compute the pre-images of 0 and 1 under this map, and explain why the pre-image of 1 is not contractible.

Exercise 6.11. Use Proposition 6.12 to prove that if

$$f\colon Y \twoheadrightarrow X$$

is a trivial fibration, then for every vertex $x \in X_0$, the fiber

$$f^{-1}(x)$$

is a contractible Kan complex.

Exercise 6.12. Use the case $n = 0$ of the lifting characterization in Proposition 6.12 to show that a trivial fibration

$$f: Y \twoheadrightarrow X$$

is surjective on 0-simplices.

Exercise 6.13. Let

$$f: Y \twoheadrightarrow X$$

be a trivial fibration, and let $y_0, y_1 \in Y_0$ be vertices such that

$$f(y_0) = f(y_1) = x.$$

Use the case $n = 1$ of the lifting characterization in Proposition 6.12 to construct a 1-simplex in the fiber $f^{-1}(x)$ connecting y_0 and y_1.

Exercise 6.14. Prove Remark 6.11: a map

$$K \to \mathbb{D}^0$$

is a trivial fibration if and only if K is a contractible Kan complex.

Exercise 6.15. Prove Lemma 6.13. That is, assume that

$$\begin{tikzcd} W \arrow[r] \arrow[d, two heads, "q"', "\simeq"] & Y \arrow[d, two heads, "p", "\simeq"'] \\ Z \arrow[r] & X \end{tikzcd}$$

is a pullback square of simplicial sets and that p is a trivial fibration. Use the lifting characterization in Proposition 6.12 to prove that q is a trivial fibration.

Exercise 6.16. Prove Lemma 6.14. That is, prove that the composition of trivial fibrations is again a trivial fibration.

Exercise 6.17. Let $f: Y \twoheadrightarrow X$ be a trivial fibration and let

$$g: Z \to X$$

be any map of simplicial sets. Prove that the projection

$$Z \times_X Y \to Z$$

is a trivial fibration.

Exercise 6.18. Let K be a Kan complex and let $L \subseteq K$ be a contractible sub-Kan complex. Show that L defines a homotopically unique homotopical property on K.

Exercise 6.19. Let $f: Y \twoheadrightarrow X$ be a Kan fibration between Kan complexes. Use Proposition 6.12 to show that f is a homotopy equivalence if and only if for every vertex x in X_0, the fiber

$$f^{-1}(x)$$

is contractible.

Exercise 6.20. Let

$$f: Y \to X$$

be a trivial Reedy fibration of simplicial spaces. Show that for every $n \geq 0$, the map of simplicial sets

$$f_n: Y_n \to X_n$$

is a homotopy equivalence. Explain why this is only one part of the definition of a trivial Reedy fibration.

Exercise 6.21. Let S be a set, regarded as a constant simplicial set. Determine exactly when S is contractible as a Kan complex.

Exercise 6.22. Let $S = \{a, b\}$ be regarded as a constant simplicial set. List all sub-Kan complexes of S. Which of them are contractible?

Exercise 6.23. Let
$$f\colon S \to T$$
be a function of sets, regarded as a map of constant simplicial sets. Compute the fiber
$$f^{-1}(t)$$
over an element $t \in T$. Determine when this fiber is contractible.

Exercise 6.24. Let
$$f\colon S \to T$$
be a map of constant simplicial sets. Use Lemma 4.26 and Proposition 6.12 to show that f is a trivial fibration if and only if the original function of sets
$$S \to T$$
is a bijection.

Exercise 6.25. Let
$$f\colon S \to T$$
be a map of constant simplicial sets. Analyze the boundary lifting condition
$$\begin{tikzcd} \partial\mathbb{D}^n \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ \mathbb{D}^n \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$
for $n = 0$ and $n = 1$. Explain how these two cases detect surjectivity and injectivity of the function $S \to T$.

Exercise 6.26. Compute the fibers of the map
$$0\colon \mathbb{D}^0 \to N(I(1))$$
over the two vertices $0, 1 \in N(I(1))_0$. Then explain why this map is a homotopy equivalence but not a trivial fibration.

Exercise 6.27. Let $K = \mathbb{D}^0$. Compute all sub-Kan complexes of K. Which of them define homotopically unique properties?

Exercise 6.28. Let $K = N(I(1))$. Consider the sub-simplicial set generated by the vertex 0:
$$\overline{\{0\}} \subseteq N(I(1)).$$
Compute this sub-simplicial set and decide whether it is a contractible sub-Kan complex.

Exercise 6.29. Let $K = N(I(1))$. Consider the sub-simplicial set generated by both vertices:
$$\overline{\{0,1\}} \subseteq N(I(1)).$$
Compute this sub-simplicial set and decide whether it is contractible.

Exercise 6.30. Let $S = \{a, b, c\}$ and $T = \{0, 1\}$ be regarded as constant simplicial sets, and let
$$f\colon S \to T$$
be given by
$$f(a) = 0, \qquad f(b) = 0, \qquad f(c) = 1.$$
Compute the fibers of f. Which fibers are contractible? Is f a trivial fibration?

Exercise 6.31. Let $S = \{a, b, c\}$ and $T = \{0, 1\}$ be regarded as constant simplicial sets, and let
$$f\colon S \to T$$
be given by
$$f(a) = 0, \qquad f(b) = 1, \qquad f(c) = 1.$$
Construct an explicit boundary lifting problem with $\partial\mathbb{D}^1 \hookrightarrow \mathbb{D}^1$ that has no lift.

7. Segal Spaces

Let us recall the diagram of a simplicial space, as depicted in Remark 3.3. In Section 5, we introduced Reedy fibrancy to ensure that "vertically" our simplicial space behaves like a space. In this section we want to ensure that "horizontally" our simplicial space behaves like a category. We recall from Theorem 2.15 that the key property was the Segal condition. Hence, we simply introduce the homotopical analogue to the Segal condition. We then observe how the resulting objects, Segal spaces, already exhibit many categorical features. Here, we fundamentally rely on the material in [Rez01, Section 5], even though some exposition differs.

Recall from Notation 3.10 the simplicial space $\mathrm{Sp}^n = \mathrm{disc}(\mathrm{Sp}^n)$ that represents the spine of $\Delta^n = \mathrm{disc}(\mathbb{D}^n)$. We now have the following result.

Lemma 7.1. *Let X be a simplicial space and $n \geq 2$. Then there is a natural isomorphism of simplicial sets*

$$\mathrm{Map}(\mathrm{Sp}^n, X) \cong X_1 \times^{1^*,0^*}_{X_0} \ldots \times^{1^*,0^*}_{X_0} X_1,$$

where there are n factors of X_1 and the pullbacks are taken over the maps $0^*, 1^* \colon X_1 \to X_0$.

Proof. Exercise 7.7 □

Previously, the Segal condition on a simplicial set gave us strict compositions. In the context of higher categories, we should generalize this condition.

Definition 7.2 (Segal space)**.** A *Segal space* is a Reedy fibrant simplicial space T such that for every $n \geq 2$ the Kan fibration

$$i_n^* \colon T_n \cong \mathrm{Map}(\Delta^n, T) \twoheadrightarrow \mathrm{Map}(\mathrm{Sp}^n, T) \cong T_1 \times^{1^*,0^*}_{T_0} \ldots \times^{1^*,0^*}_{T_0} T_1$$

is a trivial Kan fibration.

Remark 7.3. Here the Reedy fibrancy of T ensures that maps from Sp^n to T correspond homotopy invariantly to compatible tuples in T_1, as explained in Lemma 4.12.

Remark 7.4. Let us note here that we can characterize Segal spaces also with lifting conditions, analogous to Definition 5.1. Namely, a Reedy fibrant simplicial space T is a Segal space, if for all $n \geq 2, k \geq 0$, the following diagram admits a lift

$$\begin{tikzcd} \mathrm{Sp}^n \times \mathbb{D}^k \coprod_{\mathrm{Sp}^n \times \partial\mathbb{D}^k} \Delta^n \times \partial\mathbb{D}^k \arrow[r] \arrow[d] & T \\ \Delta^n \times \mathbb{D}^k \arrow[ur, dashed] & \end{tikzcd} .$$

The proof follows from understanding how lifting conditions interact with products and hom-sets. See [Rez22, Section 21], and particularly [Rez22, Proposition 21.5], for an elegant introduction to this concept and a proof.

Finally, analogous to Proposition 2.9, we have the following unified characterization of the Segal condition via mapping simplicial spaces

Proposition 7.5. *Let T be a Reedy fibrant simplicial space. Then the following are equivalent:*

(1) T is a Segal space.
(2) The map $i_2^* \colon [\![\Delta^2, T]\!] \twoheadrightarrow [\![\mathrm{Sp}^2, T]\!]$ is a trivial Reedy fibration.

Proof. Exercise 7.8 □

Intuition 7.6. As we saw in Section 6, being a trivial fibration is a homotopical analogue of dependent uniqueness. Hence, i_2^* in Proposition 7.5 being an equivalence means "the simplicial space of lifts is homotopically unique".

Hence, the statement of Proposition 7.5 can be interpreted as, "the space of lifts of i_n is homotopically unique for all $n \geq 2$ if and only if the simplicial space of lifts of i_2 is homotopically unique"

Remark 7.7. While the statement "the simplicial space of lifts is homotopically unique" seems very suited to our situation at hand, articulating is not very straightforward in a set-theoretic context. Ideally an alternative foundation is more suited, which would allow setting up the Segal condition in a more straightforward manner.

We now commence with the category theory of Segal spaces.

Definition 7.8 (Object of a Segal space)**.** Let T be a Segal space. We define the *objects of* T as the set of vertices in T_0. Thus, $\mathrm{Obj}_T = T_{00}$.

Remark 7.9. This definition is consistent with our characterization of nerves of categories, where $N\mathcal{C}_0$ is precisely the set of objects (Remark 2.2).

In a category $\mathcal{C}$, given two objects x, y, we have a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$. In the higher categorical realm, we expect more structure.

Definition 7.10 (Morphism of a Segal space)**.** Let T be a Segal space. A *morphism* in T is a point in T_1, so an element in T_{10}.

For each morphism f, 0^*f is the *source* and 1^*f is the *target*. For two objects x, y in T, we commonly use the notation $f\colon x \to y$, for a morphism f from x to y.

Definition 7.11 (Mapping space of a Segal space)**.** For two objects x, y in T, we define the *mapping space* by the following pullback:

$$\begin{tikzcd} \mathrm{map}_T(x,y) \arrow[r] \arrow[d, two heads] & T_1 \arrow[d, two heads, "(0^*,1^*)"] \\ \mathbb{D}^0 \arrow[r, "(x,y)"] & T_0 \times T_0 \end{tikzcd}$$

or in other words the fiber of $(0^*, 1^*)$ over the point (x, y).

Notice we can actually call it a space, as we have a Kan complex.

Lemma 7.12. *Let T be a Segal space and let x, y be two objects. Then the mapping space $\mathrm{map}_T(x, y)$ is a Kan complex.*

Proof. Exercise 7.9 □

Remark 7.13. Recall from Remark 2.16 that for a category $\mathcal{C}$, we recover the hom set via an analogous pullback. This provides further evidence that this is a good way to define the mapping space in a Segal space.

Up until now we have not used the Segal condition in any way. However, we want to use it to define composition.

Intuition 7.14 (Composition via the Segal condition for n=2). The first condition states that the map $T_2 \xrightarrow{\simeq} T_1 \times_{T_0}^{1^*,0^*} T_1$ is a trivial Kan fibration. Note that T_2 is the space of 2-cells. Concretely, we depict a 2-cell σ as follows:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

Similarly, we think of $T_1 \times_{T_0}^{1^*,0^*} T_1$ as the space of *two composable arrows*, which we depict as:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

The Segal condition states that every such diagram can be filled out to a complete 2-cell:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & {} & z \end{tikzcd}$$

From this point of view, we think of h as the composition of f and g, and we think of σ as a witness of that composition. Thus we often depict h as $g \circ f$.

Based on this intuition we now have the following definition.

Definition 7.15 (Composition in a Segal space)**.** Let T be a Segal space, let x, y, z be objects, and let $f: x \to y$ and $g: y \to z$ be two morphisms. A *composition* is a choice of lift

$$\begin{tikzcd} & \mathrm{Sp}^2 \arrow[r, "{(f,g)}"] \arrow[d] & T \\ \Delta^1 \arrow[r, "02"'] \arrow[urr, dashed, "g\circ f"] & \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Now if we have three morphisms $f: x \to y, g: y \to z, h: z \to w$, we might hope that the two compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ coincide. Let us observe an intuitive argument why this should be the case.

Intuition 7.16 (Associativity via the Segal condition for n=3)*.* The second condition states that the map $T_3 \overset{\simeq}{\twoheadrightarrow} T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1$ is a trivial Kan fibration. Here T_3 is the space of 3-cells, which we can depict as a tetrahedron.

$$\begin{tikzcd} & w & \\ x \arrow[ur] \arrow[rr, "f"] \arrow[dr] & \arrow[r, phantom, "\gamma"] \arrow[d, phantom, "\sigma"] & y \arrow[ul] \arrow[dl, "g"] \\ & z \arrow[uu, "h"] & \end{tikzcd}$$

where all 2-cells and the middle 3-cell are filled out. On the other hand $T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1$ can be depicted as three composable arrows.

$$\begin{tikzcd} & w & \\ x \arrow[rr, "f"] & & y \arrow[dl, "g"] \\ & z \arrow[uu, "h"] & \end{tikzcd}$$

The Segal condition states that every such diagram can be filled out to a complete 3-cell:

$$\begin{tikzcd} & w & \\ x \arrow[ur, dashed, bend left=10, "(h\circ g)\circ f"] \arrow[ur, dashed, bend right=10, "h\circ(g\circ f)"'] \arrow[rr, "f"] \arrow[dr, dashed, "g\circ f"'] & & y \arrow[ul, dashed, "h\circ g"'] \arrow[dl, "g"] \\ & z \arrow[uu, "h"] & \end{tikzcd}$$

From the point of view of Intuition 7.14, the 1-cell from x to w is both a choice of composition $(h \circ g) \circ f$ and $h \circ (g \circ f)$.

Intuition 7.14 and Intuition 7.16 are first indications of how a Segal space has categorical features. We now want to make these intuitive ideas into a precise argument. This requires generalized mapping spaces, the space of compositions, and homotopic morphisms.

Definition 7.17 (Homotopic morphisms in a Segal space)**.** Let T be a Segal space, let x, y be two objects and let f, g in $\mathrm{map}_T(x, y)$ be two morphisms. We say f and g are *homotopic* if there is a map $\mathbb{D}^1 \to \mathrm{map}_T(x, y)$ sending 0 to f and 1 to g.

Notation 7.18. If $f, g\colon x \to y$ are two morphisms in a Segal space T that are homotopic, we write $f \simeq g$.

Our notation is justified by the following basic observation.

Lemma 7.19. *Let T be a Segal space and let x, y be two objects. Then the relation $\simeq$ on $\mathrm{map}_T(x, y)$ is an equivalence relation.*

Proof. Exercise 7.10 □

Definition 7.20 (Generalized mapping space of a Segal space)**.** Let T be a Segal space and let $x_0, x_1, ..., x_n$ be objects in T. The *generalized mapping space* $\mathrm{map}_T(x_0, ..., x_n)$ is defined as the following pullback:

$$\begin{tikzcd}
\mathrm{map}_T(x_0, ..., x_n) \arrow[r] \arrow[d, two heads] & T_n \arrow[d, two heads, "(0^*, 1^*, ..., n^*)"] \\
\mathbb{D}^0 \arrow[r, "(x_0, x_1, ..., x_n)"] & (T_0)^{n+1}.
\end{tikzcd}$$

We now have the following result analogous to Lemma 7.12.

Lemma 7.21. *Let T be a Segal space and let $x_0, x_1, ..., x_n$ be objects in T. Then $\mathrm{map}_T(x_0, ..., x_n)$ is a Kan complex.*

Proof. Exercise 7.11 □

Again, up to here we have not used the Segal condition. However, using the general Segal condition we have the following result about generalized mapping spaces and mapping spaces.

Proposition 7.22. *Let T be a Segal space and let $x_0, x_1, ..., x_n$ be objects in T, where $n \geq 2$. Then the trivial Kan fibration given in Definition 7.2 induces a trivial Kan fibration*

$$\mathrm{map}_T(x_0, ..., x_n) \overset{\simeq}{\twoheadrightarrow} \mathrm{map}_T(x_0, x_1) \times ... \times \mathrm{map}_T(x_{n-1}, x_n).$$

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{D}^0 \arrow[r, "(x_0, x_1, ..., x_n)"] \arrow[d, equal] & (T_0)^{n+1} \arrow[d, equal] & T_n \arrow[l, two heads] \arrow[d, two heads, "\simeq"] \\
\mathbb{D}^0 \arrow[r, "(x_0, x_1, ..., x_n)"] & (T_0)^{n+1} & T_1 \times_{T_0} ... \times_{T_0} T_1 \arrow[l, two heads]
\end{tikzcd} .$$

Here all vertical maps are homotopy equivalences. So, by Remark 4.13, the induced map on pullbacks will be a homotopy equivalence as well. By Definition 7.20, the pullback of the top row is precisely $\mathrm{map}_T(x_0, ..., x_n)$. Thus we are left with computing the pullback of the bottom row.

Let us denote the pullback of the bottom diagram by P_{n+1}. We proceed by induction on n. If $n = 1$, then by Definition 7.11, $P_{1+1} = P_2 = \mathrm{map}_T(x_0, x_1)$. Now assuming the result holds for

n, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{D}^0 \arrow[r, "(x_0,\ldots,x_{n-1})"] \arrow[d, equal] & T_0^n \arrow[d, "\pi_n"] & T_1 \times_{T_0}^{1^*,0^*} \ldots \times_{T_0}^{1^*,0^*} T_1 \arrow[l, two heads, "(0^*,\ldots,(n-1)^*)"'] \arrow[d, "1^*\pi_{n-1}"] & P_n \arrow[d] \\
\mathbb{D}^0 \arrow[r, "x_{n-1}"] \arrow[d, equal] & T_0 \arrow[r, equal] & T_0 \arrow[r, squiggly] & \mathbb{D}^0 \\
\mathbb{D}^0 \arrow[r, "(x_{n-1},x_n)"] & T_0 \times T_0 \arrow[u, "\pi_1"] \arrow[d, squiggly] & T_1 \arrow[l, two heads, "(0^*,1^*)"'] \arrow[u, "0^*"] & \mathrm{map}_T(x_{n-1},x_n) \arrow[u] \\
\mathbb{D}^0 \arrow[r, "(x_0,\ldots,x_n)"] & T_0^{n+1} & T_1 \times_{T_0} \ldots \times_{T_0} T_1 \arrow[l, two heads, "(0^*,\ldots,n^*)"'] &
\end{tikzcd}.$$

Taking the pullback of each column gives us the diagram below it, which is the desired diagram, whereas taking the pullback of each row gives us the diagram to the right. By pullback commutativity, the pullbacks of these two resulting diagrams coincide, meaning we have equivalences

$$P_{n+1} \xrightarrow{\simeq} P_n \times \mathrm{map}_T(x_{n-1}, x_n) \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} \mathrm{map}_T(x_0, x_1) \times \ldots \times \mathrm{map}_T(x_{n-1}, x_n),$$

finishing the result. □

We can now use these results to define and study the space of compositions.

Definition 7.23 (Space of compositions in a Segal space)**.** Let T be a Segal space, let x, y, z be three objects, and let $f: x \to y, g: y \to z$ be two morphisms. The *space of compositions* of f and g is defined by the following pullback:

$$\begin{tikzcd}
\mathrm{Comp}_T(f,g) \arrow[r] \arrow[d, two heads] & \mathrm{map}_T(x,y,z) \arrow[d, two heads, "(01^*,12^*)"] \\
\mathbb{D}^0 \arrow[r, "(f,g)"] & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z).
\end{tikzcd}$$

Proposition 7.24. *Let T be a Segal space, let x, y, z be three objects, and let $f: x \to y, g: y \to z$ be two morphisms. Then the space of compositions $\mathrm{Comp}_T(f,g)$ is a contractible Kan complex.*

Proof. Exercise 7.13 □

Intuition 7.25. A point in $\mathrm{Comp}_T(f,g)$ is precisely a choice of composition of f and g, i.e. a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z
\end{tikzcd}.$$

From this perspective, Proposition 7.24 is telling us that the composition of f and g is *homotopically* unique, in the sense of Section 6.

We can use this contractibility result to show any two choices of compositions in the target mapping space are homotopic.

Lemma 7.26. *Let T be a Segal space, let x, y, z be three objects, and let $f: x \to y, g: y \to z$ be two morphisms. Then any two choices of compositions of f and g are homotopic in $\mathrm{map}_T(x,z)$.*

Proof. Let $h, h' \colon x \to z$ be two choices of compositions of f and g. Let $\sigma, \sigma' \colon \mathbb{D}^0 \to \mathrm{Comp}_T(f,g)$ be witnesses for the compositions h, h', respectively. We have the following diagram

$$\begin{tikzcd} \partial\mathbb{D}^1 \arrow[r, "\sigma+\sigma'"'] \arrow[d] \arrow[rrr, bend left, "h+h'"] & \mathrm{Comp}_T(f,g) \arrow[r] \arrow[d, "\simeq"] & \mathrm{map}_T(x,y,z) \arrow[r, "02^*"'] \arrow[d, "\simeq"] & \mathrm{map}_T(x,z) \\ \mathbb{D}^1 \arrow[r] \arrow[ur, dashed, "H"] & \mathbb{D}^0 \arrow[r, "(f,g)"] & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) & \end{tikzcd},$$

which lifts by Proposition 6.12. The composition map $\mathbb{D}^1 \to \mathrm{map}_T(x,z)$ gives us the desired homotopy between h and h'. □

In fact we can use this proof method to show a related, and very important, result.

Lemma 7.27. *Let T be a Segal space, let x, y, z be three objects, and let $f, f' \colon x \to y, g, g' \colon y \to z$ be four morphisms. Moreover, let H_f be a homotopy from f to f', and let H_g be a homotopy from g to g'. Then any chosen composition of f and g is homotopic to any chosen composition of f' and g'.*

Proof. Let $h, h' \colon x \to z$ be choices of compositions of f, g and f', g', respectively. Moreover, let $\sigma, \sigma' \colon \mathbb{D}^0 \to \mathrm{map}_T(x,y,z)$ be witnesses for h, h', respectively. We have the following diagram

$$\begin{tikzcd} \partial\mathbb{D}^1 \arrow[r, "\sigma+\sigma'"'] \arrow[d] \arrow[rr, bend left, "h+h'"] & \mathrm{map}_T(x,y,z) \arrow[r, "02^*"'] \arrow[d, "\simeq"] & \mathrm{map}_T(x,z) \\ \mathbb{D}^1 \arrow[r, "(H_f,H_g)"'] \arrow[ur, dashed, "H"] & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) & \end{tikzcd},$$

which lifts by Proposition 6.12. The composition map $\mathbb{D}^1 \to \mathrm{map}_T(x,z)$ gives us the desired homotopy between the two compositions. □

We can use these observations to construct "pre-composition" and "post-composition" maps.

Lemma 7.28. *Let T be a Segal space, let x, y, z be three objects, and let $f \colon x \to y$ be a morphism. Then there are maps of Kan complexes*

- $f^* \colon \mathrm{map}_T(y,z) \to \mathrm{map}_T(x,z)$ *sending g to $g \circ f$*
- $f_* \colon \mathrm{map}_T(z,x) \to \mathrm{map}_T(z,y)$ *sending g to $f \circ g$*

Proof. We consider the first case and the second one is Exercise 7.14. We have the following diagram

$$\begin{tikzcd} \mathrm{map}_T(y,z) \arrow[r, "\{f\}\times\mathrm{id}"] & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \arrow[r, dashed, bend left] & \mathrm{map}_T(x,y,z) \arrow[l, two heads, "\simeq"] \arrow[r] & \mathrm{map}_T(x,z) \end{tikzcd}.$$

By Proposition 7.22, there is a homotopy inverse map $\mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \to \mathrm{map}_T(x,y,z)$. Composing all these morphisms gives us a map $\mathrm{map}_T(y,z) \to \mathrm{map}_T(x,z)$, which by definition pre-composes with f. □

Remark 7.29. We should note here the maps f^*, f_* defined in Lemma 7.28 depend on a concrete choice of homotopy inverse for the given trivial Kan fibration, which generally cannot be chosen in a canonical or natural manner.

Lemma 7.30. *Let T be a Segal space, let x, y, z be three objects, and let $f, g \colon x \to y$ be two morphisms. If $f \simeq g$, then $f^* \simeq g^*$ and $f_* \simeq g_*$.*

Proof. We consider the case of pre-composition and the other case is Exercise 7.15. Let $H \colon \mathbb{D}^1 \to \mathrm{map}_T(x,y)$ be a homotopy from f to g. We have the following diagram

$$\begin{tikzcd} \mathbb{D}^1 \times \mathrm{map}_T(y,z) \arrow[r, "H\times\mathrm{id}"] & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \arrow[r, dashed, bend left] & \mathrm{map}_T(x,y,z) \arrow[l, two heads, "\simeq"] \arrow[r] & \mathrm{map}_T(x,z) \end{tikzcd}.$$

Again, choosing a homotopy inverse for the middle map and composing gives us a map of simplicial sets $\mathbb{D}^1 \times \mathrm{map}_T(y,z) \to \mathrm{map}_T(x,z)$, which by definition is a homotopy between f^* and g^*. □

We can now move on to further aspects of the category theory of Segal spaces, namely identity morphisms and associativity.

Definition 7.31 (Identity map in a Segal space)**.** Let T be a Segal space and let x be an object in T. The *identity map* of x is the morphism $\mathrm{id}_x = 00^*(x)$ in T_1.

Lemma 7.32. *Let T be a Segal space and let x be an object in T. Then id_x has source and target x.*

Proof. Exercise 7.16 □

We now have the following result.

Proposition 7.33. *Let T be a Segal space, let x, y, z, w be four objects. Let $f\colon x \to y$, $g\colon y \to z$ and $h\colon z \to w$. Then, $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are homotopic, and both $\mathrm{id}_y \circ f$, $f \circ \mathrm{id}_x$ are homotopic to f. This means composition is homotopically associative and unital.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \times \mathrm{map}_T(z,w) & \\
 & \mathrm{map}_T(x,y,z,w) \arrow[u, "\simeq"] \arrow[dl, two heads, "013^*"'] \arrow[dr, two heads, "023^*"] & \\
\mathrm{map}_T(x,y,w) \arrow[dr, two heads, "02^*"'] & & \mathrm{map}_T(x,z,w) \arrow[dl, two heads, "02^*"] \\
 & \mathrm{map}_T(x,w) &
\end{tikzcd}$$

Let us take a point (f, g, h) in $\mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \times \mathrm{map}_T(z,w)$. Via the equivalence, we can lift it to an element σ in $\mathrm{map}_T(x,y,z,w)$. Going through the left-hand map gives us $(h \circ g) \circ f$, but the right-hand map gives us $h \circ (g \circ f)$. This proves associativity.

For the identity relation, let $f\colon x \to y$. This gives us a 2-cell $001^*(f)$ in $\mathrm{map}_T(x,x,y)$, by Exercise 7.17 this satisfies:

- $01^*001^*(f) = \mathrm{id}_x$,
- $02^*001^*(f) = f$,
- $12^*001^*(f) = f$.

This proves that composition of f and id_x is f. The other case is Exercise 7.18. □

Identity and associativity help us better understand the properties of the pre-composition and the post-composition maps on mapping spaces.

Lemma 7.34. *Let T be a Segal space and let x, y be two objects. Then $\mathrm{id}_x^* \simeq \mathrm{id}_{\mathrm{map}_T(x,y)} \simeq (\mathrm{id}_y)_*$.*

Proof. We only prove the first equivalence and the second one is Exercise 7.19. For a morphism f, $\mathrm{id}_x^*(f) = f \circ \mathrm{id}_x \simeq f = \mathrm{id}_{\mathrm{map}_T(x,y)}(f)$. □

Lemma 7.35. *Let T be a Segal space, let x, y, z be three objects, and let $f\colon x \to y$, $g\colon y \to z$ be two morphisms. Then $f^* \circ g^* \simeq (g \circ f)^*$ and $g_* \circ f_* \simeq (g \circ f)_*$.*

Proof. We prove the case of pre-composition and the other case is Exercise 7.20 . For a given object w and morphism $h\colon z \to w$, we have

$$f^* \circ g^*(h) \simeq f^*(h \circ g) \simeq (h \circ g) \circ f \simeq h \circ (g \circ f) \simeq (g \circ f)^*(h),$$

where we used the definition of f^*, g^* and associativity (Proposition 7.33). □

Let us move on to functors of Segal spaces.

Definition 7.36 (Functor of Segal spaces)**.** Let T and U be two Segal spaces. A *functor* $F\colon T \to U$ is a map of simplicial spaces.

On the surface, this seems to differ from the classical definition of functors between categories. However, we have the following result.

Proposition 7.37. *Let $F\colon T \to U$ be a functor between Segal spaces. Then F induces the following:*
- *A function $\mathrm{Obj}_T \to \mathrm{Obj}_U$ between the objects of T and U.*
- *For every two objects x, y in T, a map of mapping spaces $\mathrm{map}_T(x, y) \to \mathrm{map}_U(Fx, Fy)$.*
- *For every three objects x, y, z in T and morphisms $f\colon x \to y$, $g\colon y \to z$, a homotopy between $F(g \circ f)$ and $Fg \circ Ff$.*
- *For every object x in T, a homotopy between $F(\mathrm{id}_x)$ and id_{Fx}.*

Proof. First, F induces a function $F_{00}\colon \mathrm{Obj}_T = T_{00} \to U_{00} = \mathrm{Obj}_U$. Next, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{D}^0 \arrow[r, "{(x,y)}"] \arrow[d, equal] & T_0 \times T_0 \arrow[d, "F_0 \times F_0"] & T_1 \arrow[l, two heads] \arrow[d, "F_1"] \\
\mathbb{D}^0 \arrow[r, "{(Fx,Fy)}"] & U_0 \times U_0 & U_1 \arrow[l, two heads]
\end{tikzcd} .$$

As the mapping spaces are just the pullbacks of these diagrams (Definition 7.11), the existence of a map $F_1\colon \mathrm{map}_T(x, y) \to \mathrm{map}_U(Fx, Fy)$ follows from the property of limits. Next, let σ be a point in $\mathrm{Comp}(f, g)$, meaning $01^*\sigma = f$, $12^*\sigma = g$. Then, by the simplicial identities $F_2\sigma$ is a point in $\mathrm{Comp}(F_1 f, F_1 g)$, as we have $01^* \circ F_2\sigma = F_1 \circ 01^*\sigma = F_1 f$, $12^* \circ F_2\sigma = F_1 \circ 12^*\sigma = F_1 g$. This gives us the desired homotopy between $F_1(02^*\sigma) = F_1(g \circ f)$ and $F_1 g \circ F_1 f$, as any two possible compositions of $F_1 f$ and $F_1 g$ are homotopic (Lemma 7.26). Finally, for a given object x in T, by Exercise 7.21, $F_1(\mathrm{id}_x) = \mathrm{id}_{F_0 x}$. □

With this result at hand, we can also define fully faithful functors of Segal spaces.

Definition 7.38 (Fully faithful functor of Segal spaces)**.** Let $F\colon T \to U$ be a functor between Segal spaces. We say F is *fully faithful* if for every two objects x, y in T, the induced map of mapping spaces $\mathrm{map}_T(x, y) \to \mathrm{map}_U(Fx, Fy)$ is a homotopy equivalence of Kan complexes.

Having defined functors between Segal spaces, we can also consider homotopies and equivalences.

Definition 7.39 (Homotopy of functors of Segal spaces)**.** Let $F, G\colon T \to U$ be two functors between Segal spaces. A *homotopy* $H\colon F \to G$ is a map of simplicial spaces $H\colon T \times \mathbb{D}^1 \to U$ such that $H|_{T\times 0^*} = F$ and $H|_{T\times 1^*} = G$.

Definition 7.40 (Equivalence of Segal spaces)**.** Let $F\colon T \to U$ be a functor between Segal spaces. We say F is an *equivalence* if there exists a functor $G\colon U \to T$, such that $G \circ F$ is homotopic to id_T and $F \circ G$ is homotopic to id_U.

Lemma 7.41. *Let T and U be Segal spaces and let*

$$F\colon T \to U$$

be an equivalence of Segal spaces. Prove that for every $n \geq 0$, the map

$$F_n\colon T_n \to U_n$$

is a homotopy equivalence of Kan complexes.

Proof. Exercise 7.22 □

We end this section with the observation that every Segal space has an associated homotopy category.

Construction 7.42. Let T be a Segal space. We now construct a directed graph with a composition operation and identities. This construction is crucial when we look at the *homotopy category* (Theorem 7.43). Let $\mathrm{Ho}(T)$ consist of the following data:

- **Objects:** The objects of $\mathrm{Ho}(T)$ are the objects of T.
- **Morphisms:** For two objects x, y in $\mathrm{Ho}(T)$ we define the set of morphisms as
$$\mathrm{Hom}_{\mathrm{Ho}(T)}(x,y) := \mathrm{map}_T(x,y)/\sim,$$
where $\sim$ is the homotopy relation defined in Definition 7.17 and Lemma 2.41.
- **Composition:** For three objects x, y, z in $\mathrm{Ho}(T)$, the composition map
$$\mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \to \mathrm{map}_T(x,z)$$
gives us a composition map
$$\mathrm{Hom}_{\mathrm{Ho}(T)}(x,y) \times \mathrm{Hom}_{\mathrm{Ho}(T)}(y,z) \to \mathrm{Hom}_{\mathrm{Ho}(T)}(x,z),$$
because homotopy respects composition, by Lemmas 7.26 and 7.27.

This construction gives us the following theorem.

Theorem 7.43 (Homotopy category of a Segal space)**.** *Let T be a Segal space. Then $\mathrm{Ho}(T)$ is a category, called the* homotopy category *of T.*

Proof. We only need to verify that the composition operation is well-defined, associative and unital. However, this is the statement of Lemmas 7.26 and 7.27 and Proposition 7.33. □

This theorem allows us to construct an ordinary category out of every Segal space, confirming the connection between Segal spaces and categories. This connection also extends to functors.

Theorem 7.44. *Let $F: T \to U$ be a functor between Segal spaces. Then F induces a functor between homotopy categories $\mathrm{Ho}(F): \mathrm{Ho}(T) \to \mathrm{Ho}(U)$.*

Proof. Exercise 7.23 □

This relation also extend to key properties.

Proposition 7.45. *Let $F: T \to U$ be a fully faithful functor between Segal spaces. Then $\mathrm{Ho}(F): \mathrm{Ho}(T) \to \mathrm{Ho}(U)$ is a fully faithful functor between homotopy categories.*

Proof. Exercise 7.24 □

Exercises.

Exercise 7.1. Let T be a Reedy fibrant simplicial space. Unpack the Segal space condition for $n = 3$. In particular, describe the map
$$i_3^*: T_3 \to T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1$$
in terms of the three edges
$$01^*, \qquad 12^*, \qquad 23^*.$$

Exercise 7.2. Let T be a Segal space and let
$$x_0, x_1, \ldots, x_n$$
be objects of T. Unpack the definition of the generalized mapping space
$$\mathrm{map}_T(x_0, \ldots, x_n).$$
What are its vertices when $n = 1, 2$?

Exercise 7.3. Let T be a Segal space and let
$$f: x \to y, \qquad g: y \to z$$
be morphisms. Unpack the definition of the space of compositions
$$\mathrm{Comp}_T(f,g).$$
Explain what data a vertex of $\mathrm{Comp}_T(f,g)$ consists of.

Exercise 7.4. Let T be a Segal space and let $f, g\colon x \to y$ be two morphisms. Explain why the statement

$$f \simeq g$$

means that f and g lie in the same path-component of the Kan complex

$$\mathrm{map}_T(x, y).$$

Exercise 7.5. Let $F\colon T \to U$ be a functor of Segal spaces. Explain why F induces maps

$$T_0 \to U_0 \qquad \text{and} \qquad T_1 \to U_1.$$

Then describe how these maps determine the function on objects and the map on morphisms.

Exercise 7.6. Let $F, G\colon T \to U$ be functors of Segal spaces. Unpack the definition of a homotopy

$$H\colon F \to G.$$

In particular, explain what the restrictions

$$H|_{T\times 0^*} = F \qquad \text{and} \qquad H|_{T\times 1^*} = G$$

mean level-wise.

Exercise 7.7. Prove Lemma 7.1, that is show that for a simplicial space X and $n \geq 2$ there is a natural isomorphism of simplicial sets

$$\mathrm{Map}(\mathrm{Sp}^n, X) \cong X_1 \times_{X_0}^{1^*,0^*} \ldots \times_{X_0}^{1^*,0^*} X_1,$$

where there are n factors of X_1 and the pullbacks are taken over the maps $0^*, 1^*\colon X_1 \to X_0$.
Hint: Lemma 1.49 and Proposition 3.20 and the proof method in Lemma 2.6 helps.

Exercise 7.8. (Hard) Prove Proposition 7.5, that is T a Reedy fibrant simplicial space the following are equivalent:

(1) T is a Segal space.
(2) The map $i_2^*\colon [\![\Delta^2, T]\!] \twoheadrightarrow [\![\mathrm{Sp}^2, T]\!]$ is a trivial Reedy fibration.

Hint: Use the definition of $[\![-,-]\!]$ to reduce the problem to the fact that the map

$$\mathrm{Map}(\Delta^n, T) \to \mathrm{Map}(\mathrm{Sp}^n, T)$$

is an equivalence for all $n \geq 2$ if and only if the map

$$\mathrm{Map}(\Delta^2 \times \Delta^n, T) \to \mathrm{Map}(\mathrm{Sp}^2 \times \Delta^n, T)$$

is an equivalence for all $n \geq 2$. Then prove this result using combinatorial methods.

Exercise 7.9. Prove Lemma 7.12, that is use Lemma 4.10 to show that for a Segal space T and objects x, y the mapping space $\mathrm{map}_T(x, y)$ is a Kan complex.

Exercise 7.10. Use Lemma 2.41 to prove Lemma 7.19: for objects x, y in a Segal space T, the relation $\simeq$ on the vertices of

$$\mathrm{map}_T(x, y)$$

is an equivalence relation.

Exercise 7.11. Prove Lemma 7.21, that is use Example 5.5 to show that for a Segal space T and objects $x_0, x_1, \ldots, x_n$ the generalized mapping space $\mathrm{map}_T(x_0, \ldots, x_n)$ is a Kan complex.

Exercise 7.12. Prove Proposition 7.22 in the case $n = 2$. Namely, show that the Segal map induces a trivial Kan fibration

$$\mathrm{map}_T(x, y, z) \xrightarrow{\simeq}\mathrel{\mkern-14mu}\rightarrow \mathrm{map}_T(x, y) \times \mathrm{map}_T(y, z).$$

Exercise 7.13. Prove Proposition 7.24: use the properties of trivial Kan fibrations and contractible Kan complexes to show that for morphisms

$$f\colon x \to y, \qquad g\colon y \to z$$

in a Segal space T, the space of compositions

$$\mathrm{Comp}_T(f,g)$$

is a contractible Kan complex.

Exercise 7.14. Prove Lemma 7.28 for post-composition. That is, let

$$f\colon x \to y$$

be a morphism in a Segal space T. After choosing a homotopy inverse for the relevant trivial Kan fibration, construct a map

$$f_*\colon \mathrm{map}_T(z,x) \to \mathrm{map}_T(z,y).$$

Exercise 7.15. Prove the post-composition part of Lemma 7.30: if

$$f,g\colon x \to y$$

are homotopic morphisms in a Segal space T, then

$$f_* \simeq g_*.$$

Exercise 7.16. Use the simplicial identities to prove Lemma 7.32: id_x, where x is an object in a Segal space T, has source and target x.

Exercise 7.17. Prove that for a morphism $f\colon x \to y$ in a Segal space T, the 2-cell $001^*(f)$ in $\mathrm{map}_T(x,x,y)$ satisfies

- $01^*001^*(f) = \mathrm{id}_x$,
- $02^*001^*(f) = f$,
- $12^*001^*(f) = f$.

Exercise 7.18. Complete the proof of Proposition 7.33: For a morphism f in a Segal spaces T, use the 2-cell $011^*(f)$ to show that the composition of id_y and f is f.

Exercise 7.19. Prove the second equivalence of Lemma 7.34: for objects x,y in a Segal space T,

$$(\mathrm{id}_y)_* \simeq \mathrm{id}_{\mathrm{map}_T(x,y)}.$$

Exercise 7.20. Prove the post-composition case of Lemma 7.35: for morphisms $f\colon x \to y, g\colon y \to z$ in a Segal space T,

$$g_* \circ f_* \simeq (g \circ f)_*.$$

Exercise 7.21. Complete the proof of Proposition 7.37: Use the simplicial identities and naturality of the map of Segal spaces $F\colon T \to U$, to show that $F_1(\mathrm{id}_x) = \mathrm{id}_{F_0 x}$, for every object x in T.

Exercise 7.22. Prove Lemma 7.41: Use the fact that $\mathbb{D}^1_n = \mathbb{D}^1$, to show that an equivalence of Segal spaces

$$F\colon T \to U$$

induces a homotopy equivalence of Kan complexes

$$F_n\colon T_n \to U_n.$$

Exercise 7.23. Prove Theorem 7.44, that is show that a functor $F\colon T \to U$ between Segal spaces induces a functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(T) \to \mathrm{Ho}(U)$ between the homotopy categories.

Exercise 7.24. Prove Proposition 7.45: show that a fully faithful functor $F\colon T \to U$ between Segal spaces induces a fully faithful functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(T) \to \mathrm{Ho}(U)$ between the homotopy categories.

Exercise 7.25. Compute Sp^3 explicitly in dimensions 0 and 1 in the horizontal direction. In particular, list the non-degenerate 0-simplices and 1-simplices of Sp^3.

Exercise 7.26. Let T be a Segal space. Compute

$$\mathrm{Map}(\mathrm{Sp}^3, T)$$

as an iterated pullback of simplicial sets. Then describe its vertices.

Exercise 7.27. Let $\mathcal{C}$ be a category and consider the Segal space

$$T = \mathrm{disc}(N\mathcal{C}).$$

Compute the objects of T, the morphisms of T, and show that for two objects $x, y \in \mathcal{C}$ the mapping space

$$\mathrm{map}_T(x, y),$$

is the constant simplicial set associated to the set

$$\mathrm{Hom}_{\mathcal{C}}(x, y).$$

Exercise 7.28. Let $\mathcal{C}$ be a category and let

$$T = \mathrm{disc}(N\mathcal{C}).$$

Compute the space of compositions

$$\mathrm{Comp}_T(f, g)$$

for two composable morphisms

$$f\colon x \to y, \qquad g\colon y \to z$$

in $\mathcal{C}$.

Exercise 7.29. Let $\mathcal{C}$ be a category. Show that

$$\mathrm{Ho}(\mathrm{disc}(N\mathcal{C})) \cong \mathcal{C}.$$

Exercise 7.30. Let $T = \Delta^2$ and consider the two morphisms

$$01\colon 0 \to 1, \qquad 12\colon 1 \to 2.$$

Compute

$$\mathrm{Comp}_T(01, 12).$$

What is the resulting composite?

Exercise 7.31. Let $T = \mathrm{disc}(NG)$, where G is a group regarded as a one-object category. Compute

$$\mathrm{Obj}_T, \qquad \mathrm{map}_T(*, *), \qquad \mathrm{Ho}(T).$$

Exercise 7.32. Let S be a discrete category, regarded as a constant simplicial set via its set of objects. Compute the Segal space

$$\mathrm{disc}(S).$$

What are its mapping spaces and its homotopy category?

Exercise 7.33. Let T be a Segal space and let $f\colon x \to y$ be a morphism. Compute the effect of

$$f^*\colon \mathrm{map}_T(y, z) \to \mathrm{map}_T(x, z)$$

on vertices. Then compute the effect of

$$f_*\colon \mathrm{map}_T(z, x) \to \mathrm{map}_T(z, y)$$

on vertices.

Exercise 7.34. Let $F\colon T \to U$ be a functor of Segal spaces and let

$$f\colon x \to y, \qquad g\colon y \to z$$

be morphisms in T. Write down explicitly the equality in $\mathrm{Ho}(U)$ expressing preservation of composition:

$$[F(g \circ f)] = [Fg \circ Ff].$$

Exercise 7.35. Let $F\colon T \to U$ be a fully faithful functor of Segal spaces. For fixed objects x, y of T, compute the induced map

$$\mathrm{Hom}_{\mathrm{Ho}(T)}(x,y) \to \mathrm{Hom}_{\mathrm{Ho}(U)}(Fx, Fy)$$

in terms of path-components of mapping spaces. Explain why it is a bijection.

8. Homotopy Equivalences in Segal Spaces

In the previous section we saw many categorical aspects of Segal spaces, such as composition, identities, and functors. In this section we will focus on the homotopical aspects of Segal spaces, in particular homotopy equivalences, which are part of the theory that distinguishes Segal spaces from ordinary categories. In particular we will present several equivalent characterizations of homotopy equivalences, and observe a variety of homotopical uniqueness results (in the sense of Section 6) regarding homotopy equivalences.

Remark 8.1. All the definitions and results presented in this section are well-known to experts. However, there does not appear to be a systematic presentation of these results in the literature, beyond the first couple of steps that can already be found in [Rez01]. As a result this section is somewhat more demanding than the rest of this note.

8.1. **Homotopy Equivalences: First Steps.**

We commence with the definition and first properties of homotopy equivalences in Segal spaces.

Definition 8.2 (Homotopy equivalence in a Segal space)**.** Let T be a Segal space and let x, y be objects in T. A morphism f in $\mathrm{map}_T(x,y)$ is a homotopy equivalence if there are maps g, h in $\mathrm{map}_T(y,x)$ such that $f \circ g$ is homotopic to id_y and $h \circ f$ is homotopic to id_x.

Remark 8.3. We can interpret the condition in Definition 8.2 as saying that id_y is a choice of composition of f and g, and, similarly, id_x is a choice of composition of h and f.

Intuition 8.4. Although it might appear that the definition only involves the existence of three maps, in reality the nature of a Segal space demands that there are several other important pieces of information. In particular, the fact that the compositions $f \circ g$ and $h \circ f$ correspond to the identity involves 2-cells that witness these compositions. The information can be captured in a diagram of the following form

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[dr, "f"] & x \\ y \arrow[u, "g"] \arrow[r, "\mathrm{id}_y"'] & y \arrow[u, "h"'] \end{tikzcd}$$

Note that, similar to the case for classical categories, inverses are homotopically unique, consistent with our philosophy (Section 6).

Lemma 8.5. *Let T be a Segal space, let x, y be objects in T, and let f in $\mathrm{map}_T(x,y)$ be a homotopy equivalence with inverses g and h. Then g and h are homotopic.*

Proof. We have the following chain of homotopies

$$\begin{aligned} g &\simeq \mathrm{id}_x \circ g && \text{Proposition 7.33} \\ &\simeq (h \circ f) \circ g && \text{Definition of } h \\ &\simeq h \circ (f \circ g) && \text{Proposition 7.33} \\ &\simeq h \circ \mathrm{id}_y && \text{Definition of } g \\ &\simeq h && \text{Proposition 7.33} \end{aligned}$$

□

Given their importance we want various alternative ways to characterize homotopy equivalences. First we observe homotopy equivalences via isomorphism in the homotopy category.

Proposition 8.6. *Let T be a Segal space and let $f\colon x \to y$ be a morphism in T. Then f is a homotopy equivalence if and only if $[f]$ is an isomorphism in* $\mathrm{Ho}(T)$*.*

Proof. The forward direction is the content of Exercise 8.4

On the other hand, if $[f]$ is an isomorphism in $\mathrm{Ho}(T)$, then there is a morphism $[g]$ in $\mathrm{Ho}(T)$ such that $[f] \circ [g] = [\mathrm{id}_y]$ and $[g] \circ [f] = [\mathrm{id}_x]$. Following Theorem 7.43, $[g] \circ [f] = [g \circ f] = [\mathrm{id}_x]$, meaning $g \circ f$ is homotopic to id_x. Similarly, $f \circ g$ is homotopic to id_y. Thus, f is a homotopy equivalence. □

Intuition 8.7. Fundamentally, the homotopy category $\mathrm{Ho}(T)$ only remembers the objects and homotopy classes of morphisms in T. Thus Proposition 8.6 can be understood as saying that "being a homotopy equivalence" does not contain any non-trivial homotopical information. See Intuition 8.44 for a more detailed discussion of this point.

Let us move on to another characterization of homotopy equivalences. In classical category theory, we can identify isomorphisms as those morphisms whose pre-composition or post-composition induces bijections on hom sets. Let us generalize that.

Proposition 8.8. *Let T be a Segal space, let x, y be two objects in T, and let $f\colon x \to y$ be a morphism. Then the following are equivalent:*

(1) f is a homotopy equivalence.
(2) For every object z, the maps $f_*\colon \mathrm{map}_T(z, x) \to \mathrm{map}_T(z, y)$ are homotopy equivalences of Kan complexes.
(3) For every object z, the maps $f^*\colon \mathrm{map}_T(y, z) \to \mathrm{map}_T(x, z)$ are homotopy equivalences of Kan complexes.

Proof. We prove (1) $\Leftrightarrow$ (2) and the case (1) $\Leftrightarrow$ (3) is Exercise 8.5. Let us assume f is a homotopy equivalence. Let us denote the right inverse of f by g, and note there is a homotopy $f \circ g \simeq \mathrm{id}_y$. Then, we have

$$\begin{aligned} f_* \circ g_* &\simeq (f \circ g)_* && \text{Lemma 7.35} \\ &\simeq (\mathrm{id}_y)_* && \text{Lemma 7.30} \\ &\simeq \mathrm{id} && \text{Lemma 7.34.} \end{aligned}$$

Using a left inverse h of f similarly gives us $h_* \circ f_* \simeq \mathrm{id}$. Thus, f_* is a homotopy equivalence with right inverse g_* and left inverse h_*.

On the other hand, let us assume f_* is a homotopy equivalence. Then, as every homotopy equivalence is a weak equivalence (Lemma 2.28), for every object z, we get a bijection of hom sets $\mathrm{Ho}([f]_*)\colon \mathrm{Hom}_{\mathrm{Ho}(T)}(z, x) \to \mathrm{Hom}_{\mathrm{Ho}(T)}(z, y)$. Thus, $[f]$ is an isomorphism in $\mathrm{Ho}(T)$, and by Proposition 8.6, f is a homotopy equivalence. □

We can use equivalences in Segal spaces to generalize various common categorical notions.

Definition 8.9 (Segal space groupoid)**.** We say a Segal space T is a *Segal space groupoid* if every morphism is a homotopy equivalence.

Lemma 8.10. *Let T be a Segal space. Then T is a Segal space groupoid if and only if* $\mathrm{Ho}(T)$ *is a groupoid.*

Proof. Exercise 8.6 □

Definition 8.11 (Essentially surjective functor of Segal spaces)**.** Let $F\colon T \to U$ be a functor between Segal spaces. We say F is *essentially surjective* if for every object y in U, there exists an object x in T such that there is a homotopy equivalence from Fx to y in U.

Lemma 8.12. *Let $F\colon T \to U$ be a functor between Segal spaces. Then F is essentially surjective if and only if* $\mathrm{Ho}(F)\colon \mathrm{Ho}(T) \to \mathrm{Ho}(U)$ *is essentially surjective.*

Proof. Exercise 8.7 □

8.2. **Homotopy Equivalences: Structure vs. Property.** We now move on to more intrinsic characterizations of homotopy equivalences, via various spaces of equivalences. We construct several spaces of equivalences, using well-chosen colimit and limit constructions.

Definition 8.13. Let $Z(3)$ be the simplicial space defined by the colimit of the following diagram.

$$\begin{tikzcd} \Delta^1 & & \Delta^1 & & \Delta^1 \\ & \Delta^0 \arrow[ul, "1"] \arrow[ur, "1"'] & & \Delta^0 \arrow[ul, "0"] \arrow[ur, "0"'] & \end{tikzcd}$$

Lemma 8.14. *Let T be a Segal space. Then there is a natural isomorphism of simplicial sets*

$$\mathrm{Map}(Z(3), T) \cong T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1.$$

Proof. Exercise 8.8 □

As a next step, we use this description to construct two maps into $\mathrm{Map}(Z(3), T)$. For this next lemma, we denote the elements in

$$(\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_1 = \{00_0, 01_0, 11_0, 12_0, 22_0, 02_0, 12_1, 02_1, 22_1\},$$

where the subscript indicates whether the element comes from the left or right copy of Δ^2.

Lemma 8.15. *Let T be a Segal space. There are natural maps*

$$02_0 + 12_0 + 02_1 \colon Z(3) \to \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2,$$

$$012 + 123 \colon \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 \to \Delta^3$$

which induce a chain of Kan fibrations

$$T_3 \overset{P_{32}}{\twoheadrightarrow} T_2 \times_{T_1}^{12^*,01^*} T_2 \overset{P_{21}}{\twoheadrightarrow} T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1.$$

Proof. We need to construct an element in the set

$$\mathrm{Hom}(\Delta^1, \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2) \underset{\mathrm{Hom}(\Delta^0, \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)}{\times} \mathrm{Hom}(\Delta^1, \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2) \underset{\mathrm{Hom}(\Delta^0, \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)}{\times} \mathrm{Hom}(\Delta^1, \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2),$$

which by the Yoneda lemma (Lemma 3.11) corresponds to an element in the set

$$(\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_1 \times_{(\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_0} (\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_1 \times_{(\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_0} (\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2)_1.$$

Such an element is given by a commutative diagram of the form

$$\begin{tikzcd} & \Delta^0 \arrow[dl, "1"'] \arrow[dr, "1"] & & \Delta^0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ \Delta^1 \arrow[drr, "02_0"'] & & \Delta^1 \arrow[d, "12_0"] & & \Delta^1 \arrow[dll, "02_1"] \\ & & \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 & & \end{tikzcd},$$

which we confirm in Exercise 8.9. Using the same argument, we get the second map $\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 \to \Delta^3$, see Exercise 8.10. via the diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "12"] \arrow[d, "01"'] & \Delta^2 \arrow[d, "012"] \\ \Delta^2 \arrow[r, "123"'] & \Delta^3 \end{tikzcd} .$$

Now, by Lemma 5.2, these maps induce Kan fibrations

$$\mathrm{Map}(\Delta^3, T) \xrightarrow{(012+123)^*} \mathrm{Map}(\Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2, T) \xrightarrow{(02_0+12_0+02_1)^*} \mathrm{Map}(Z(3), T), \ ,$$

as T is Reedy fibrant. Finally, by Proposition 3.20, this corresponds to the desired maps

$$T_3 \overset{P_{32}}{\twoheadrightarrow} T_2 \times_{T_1}^{12^*,01^*} T_2 \overset{P_{21}}{\twoheadrightarrow} T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1.$$

□

We want to make some observations about P_{32}. This requires defining a helper function.

Lemma 8.16. *Let T be a Segal space. Then there is a map*

$$01_0 + 12_0 + 12_1 \colon \Delta^1 \coprod_{\Delta^0}^{1,0} \Delta^1 \coprod_{\Delta^0}^{1,0} \Delta^1 \to \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2,$$

which induces a Kan fibration

$$C_{21} \colon T_2 \times_{T_1}^{12^*,01^*} T_2 \twoheadrightarrow T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1$$

Proof. Exercise 8.11 □

Lemma 8.17. *Let T be a Segal space. The map $C_{21} \colon T_2 \times_{T_1}^{12^*,01^*} T_2 \to T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1$ is isomorphic to the map $i_2^* \times i_2^* \colon T_2 \times_{T_1}^{12^*,01^*} T_2 \to (T_1 \times_{T_0} T_1) \times_{T_1}^{\pi_2,\pi_1} (T_1 \times_{T_0} T_1)$ and is a trivial Kan fibration.*

Proof. First we establish the isomorphism. First of all we have the following commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "12"] \arrow[d, "01"] & \mathrm{Sp}^2 \arrow[d, "01+12"] \\ \mathrm{Sp}^2 \arrow[r, "12+23"] & \mathrm{Sp}^3 \end{tikzcd} ,$$

which induces a map $(01+12)+(12+23) \colon \mathrm{Sp}^2 \coprod_{\Delta^1}^{12,01} \mathrm{Sp}^2 \to \mathrm{Sp}^3$. By Exercise 8.12, this map is an isomorphism. Next, we show in Exercise 8.13 that the following diagram commutes

$$\begin{tikzcd} \mathrm{Sp}^2 \coprod_{\Delta^1}^{12,01} \mathrm{Sp}^2 \arrow[rr, "(01+12)+(12+23)", "\cong"'] \arrow[dr, "i_2+i_2"'] & & \mathrm{Sp}^3 \arrow[dl, "01_0+01_1+12_1"] \\ & \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 & \end{tikzcd} .$$

Finally, applying $\mathrm{Map}(-, T)$ to this commutative diagram and applying Proposition 3.20, we get the desired commutative diagram

$$\begin{tikzcd} & T_2 \times_{T_1}^{12^*,01^*} T_2 \arrow[dl, two heads, "C_{21}"'] \arrow[dr, two heads, "i_2^* \times i_2^*"] & \\ T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1 \arrow[rr, "\cong"] & & (T_1 \times_{T_0}^{1^*,0^*} T_1) \times_{T_1}^{\pi_2,\pi_1} (T_1 \times_{T_0}^{1^*,0^*} T_1) \end{tikzcd} ,$$

where all morphisms are Kan fibrations, by Lemma 5.2, as T is Reedy fibrant.

We now move on to the second part. By 2-out-of-3 (Remark 2.46), to show that C_{21} is a trivial Kan fibration, it suffices to show that $i_2^* \times i_2^*$ is a homotopy equivalence. This map is the pullback of the following diagram

$$\begin{tikzcd} T_2 \arrow[r, two heads, "12^*"] \arrow[d, two heads, "\simeq"] & T_1 \arrow[d, equal] & T_2 \arrow[l, two heads, "01^*"'] \arrow[d, two heads, "\simeq"] \\ T_1 \times_{T_0} T_1 \arrow[r, two heads, "\pi_2"] & T_1 & T_1 \times_{T_0} T_1 \arrow[l, two heads, "\pi_1"'] \end{tikzcd},$$

and so the result follows from Lemma 4.12, as all horizontal maps are Kan fibrations, and the vertical maps are homotopy equivalences. $\square$

Lemma 8.18. *Let T be a Segal space. The map $P_{32}\colon T_3 \to T_2 \times_{T_1}^{12^*,01^*} T_2$ is a trivial Kan fibration.*

Proof. The argument proceeds analogously to the proof of Lemma 8.17. By Lemma 8.15, P_{32} is a Kan fibration. So, we only need to show it is a homotopy equivalence. Following Exercise 8.14 the diagram commutes

$$\begin{tikzcd} & Sp^3 \arrow[dl, "01_0+12_0+12_1"'] \arrow[dr, "01+12+23"] & \\ \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 \arrow[rr, "012+123"'] & & \Delta^3 \end{tikzcd},$$

which induces a commutative diagram

$$\begin{tikzcd} T_3 \arrow[rr, two heads, "P_{32}"] \arrow[dr, two heads] & & T_2 \times_{T_1}^{12^*,01^*} T_2 \arrow[dl, two heads, "i_2^* \times i_2^*"] \\ & T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1 & \end{tikzcd}.$$

By the Segal condition and Lemma 8.17, the diagonal maps are trivial Kan fibrations. Thus, by 2-out-of-3 (Remark 2.46), P_{32} is a homotopy equivalence of Kan complexes. $\square$

See Intuition 8.22 for some intuition about this equivalence.

Lemma 8.19. *Let T be a Segal space. There is a natural map*

$$(11^*, 01^*, 00^*)\colon T_1 \to T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1.$$

Proof. We need to construct an element in the set

$$\mathrm{Hom}(T_1, T_1) \times_{\mathrm{Hom}(T_1,T_0)} \mathrm{Hom}(T_1, T_1) \times_{\mathrm{Hom}(T_1,T_0)} \mathrm{Hom}(T_1, T_1).$$

Such an element is given by the following diagram

$$\begin{tikzcd} & & T_1 \arrow[dll, "11^*"'] \arrow[d, "01^*"] \arrow[drr, "00^*"] & & \\ T_1 \arrow[dr, "1^*"] & & T_1 \arrow[dl, "1^*"'] \arrow[dr, "0^*"] & & T_1 \arrow[dl, "0^*"] \\ & T_0 & & T_0 & \end{tikzcd},$$

which we show in Exercise 8.15. $\square$

Intuition 8.20. Intuitively, the map in Lemma 8.19 takes a morphism $f\colon x \to y$ and sends it to the triple $(\mathrm{id}_y, f, \mathrm{id}_x)$, where the pullback specifies the first morphism is the identity on the codomain of f and the third morphism is the identity on the domain of f.

We now have all the pieces in place to present another way to define homotopy equivalences.

Lemma 8.21. *Let T be a Segal space, let x, y be objects in T, and let $f\colon x \to y$ be a morphism in T. Then f is a homotopy equivalence if and only if the following diagram admits a lift H_2 or H_3*

$$\begin{tikzcd}
 & & T_3 \arrow[d, "P_{32}"] \\
 & & T_2 \times_{T_1}^{12^*,01^*} T_2 \arrow[d, "P_{21}"] \\
\mathbb{D}^0 \arrow[r, "f"] \arrow[uurr, dashed, bend left, "H_3"] \arrow[urr, dashed, bend left, "H_2"] & T_1 \arrow[r, "{(11^*,01^*,00^*)}"] & T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1
\end{tikzcd}.$$

Proof. By Lemma 8.18, the map P_{32} is a trivial fibration of Kan complexes, so it suffices to show that f is a homotopy equivalence if and only if the diagram admits a lift H_2. Let $f\colon x \to y$ be a homotopy equivalence. Denote its right inverse by g and its left inverse by h. Let $\sigma_g, \sigma_h\colon \Delta^2 \to T$ be the maps corresponding to the fact that fg composes to id_y and hf composes to id_x. Then $H_2 := (\sigma_g, \sigma_h)$ is a vertex in $T_2 \times_{T_1}^{12^*,01^*} T_2$ that lifts $(\mathrm{id}_y, f, \mathrm{id}_x)$ in $T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1$. On the other hand, assume that $(\mathrm{id}_y, f, \mathrm{id}_x)$ lifts to $H_2 = (\sigma_1, \sigma_2)$ in $T_2 \times_{T_1}^{12^*,01^*} T_2$. Let us denote $01^*\sigma_1 = g$ and $12^*\sigma_2 = h$. Then σ_1 is a witness for $fg \simeq \mathrm{id}_y$ and σ_2 is a witness for $hf \simeq \mathrm{id}_x$, meaning f is a homotopy equivalence. □

Intuition 8.22. This characterization appears technical, but is diagrammatically intuitive. The element $(\mathrm{id}_y, f, \mathrm{id}_x)$ in $T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1$ can be represented by the diagram:

$$\begin{tikzcd}
 & x & \\
y \arrow[dr, "\mathrm{id}_y"'] & & x \arrow[ul, "\mathrm{id}_x"'] \arrow[dl, "f"] \\
 & y &
\end{tikzcd}$$

A lift to an element in $T_2 \times_{T_1} T_2$ is a diagram of the following form.

$$\begin{tikzcd}
 & x & \\
y \arrow[dr, "\mathrm{id}_y"'] \arrow[rr, dashed, "g"] \arrow[r, phantom, shift right=3, "\sigma_1"] & \arrow[ur, phantom, "\sigma_2"] & x \arrow[ul, "\mathrm{id}_x"'] \arrow[dl, "f"] \\
 & y \arrow[uu, dashed, "h"] &
\end{tikzcd}$$

The data in this diagram is exactly that of two morphisms $g, h\colon y \to x$ and two 2-cells σ_1, σ_2 that give us the right and left inverses.

On the other hand, a lift to an element in T_3 is a diagram of the following form.

$$\begin{tikzcd}
 & x & \\
y \arrow[ur, dashed, "g"] \arrow[dr, "\mathrm{id}_y"'] \arrow[rr, dashed, "g"] \arrow[r, phantom, shift right=3, "\sigma_1"] & \arrow[ur, phantom, "\sigma_2"] & x \arrow[ul, "\mathrm{id}_x"'] \arrow[dl, "f"] \\
 & y \arrow[uu, dashed, "h"] &
\end{tikzcd}$$

The data in this diagram is precisely the previous data along with the data that the two morphisms $g, h\colon y \to x$ are equivalent (which is witnessed by the front 2-cell with boundaries id_y, g, h).

Hence, the fact that the existence of lifts to T_3 and $T_2 \times_{T_1}^{12^*,01^*} T_2$ is equivalent (Lemma 8.18), corresponds to the fact that if a morphism has a left inverse and a right inverse, then those inverses are homotopic, as we observed in Lemma 8.5.

This new method gives us an easy way of showing that the notion of a homotopy equivalence is homotopy invariant.

Lemma 8.23. *Let $\gamma\colon \mathbb{D}^1 \to T_1$ be a path from $\gamma(0) = f$ to $\gamma(1) = f'$ in T_1. If f' is a homotopy equivalence, then f is also a homotopy equivalence.*

Proof. By Lemma 8.21, f' admits a lift $H'\colon \mathbb{D}^0 \to T_3$. Now, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{D}^0 \arrow[rrr, "H'"] \arrow[d, "1"] & & & T_3 \arrow[d, two heads, "P_{21}\circ P_{32}"] \\
\mathbb{D}^1 \arrow[r, "\gamma"'] \arrow[urrr, dashed, "\tilde{\gamma}"] & T_1 \arrow[r, "{(11^*,01^*,00^*)}"] & & T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1
\end{tikzcd}.$$

This diagram lifts because the right-hand map is a fibration. Thus $\tilde{\gamma}(0)$ is the lift of the given map $(11^*f, f, 00^*f)$, which, again by Lemma 8.21, proves f is a homotopy equivalence. □

We now want to proceed to show that there are various spaces of equivalences, which are all homotopy equivalent to each other and give us a more intrinsic way to understand homotopy equivalences. We start with the most straightforward one.

Definition 8.24 (Space of homotopy equivalences)**.** Let T be a Segal space. Let T_{hoequiv} be the sub-simplicial set of T_1 generated by the elements in T_{10} that are homotopy equivalences (Example 1.32). We call T_{hoequiv} the *space of homotopy equivalences*.

We now immediately have the following observation about T_{hoequiv}.

Lemma 8.25. *The map $i\colon T_{\text{hoequiv}} \hookrightarrow T_1$ is an inclusion of path-components (Definition 4.22) and a Kan fibration.*

Proof. Let σ be in T_{1n}. As the vertices $0^*\sigma, ...n^*\sigma$ in T_{10} are all connected by edges in T_{11}, if one of them is a homotopy equivalence, then all of them are homotopy equivalences, by Lemma 8.23. This means that σ is in $(T_{\text{hoequiv}})_n$ if and only if $0^*\sigma$ is in T_{hoequiv}, which is equivalent to saying that $\pi_0\sigma = [0^*\sigma]$ lies in $\pi_0(T_{\text{hoequiv}})$. The argument implies that the following is a pullback diagram

$$\begin{tikzcd}
T_{\text{hoequiv}} \arrow[r, "\pi_0"] \arrow[d, two heads, "i"] \arrow[dr, phantom, "\ulcorner"] & \pi_0(T_{\text{hoequiv}}) \arrow[d, two heads, "\pi_0(i)"] \\
T_1 \arrow[r, "\pi_0"] & \pi_0(T_1)
\end{tikzcd},$$

meaning that $T_{\text{hoequiv}} \to T_1$ is an inclusion of path-components (Definition 4.22). Finally, by Proposition 4.28, it is also a Kan fibration. □

We now use P_{32}, P_{21} to pursue alternative characterizations of the space T_{hoequiv} and the map $i\colon T_{\text{hoequiv}} \to T_1$, which will give us more insight into homotopy equivalences in Segal spaces.

Definition 8.26 (Space of homotopy equivalences with (compatible) inverse)**.** Let T be a Segal space. We define the *space of homotopy equivalences with compatible inverse* T_{hoeqcomp} and the *space* *of homotopy equivalences with inverses* $T_{\text{hoeqchoice}}$ via the following pullback diagrams

$$\begin{tikzcd}
T_{\text{hoeqcomp}} \arrow[rr] \arrow[d, two heads, "P_{\text{forgcomp}}"'] \arrow[drr, phantom, "\ulcorner"] & & T_3 \arrow[d, two heads, "P_{32}"] \\
T_{\text{hoeqchoice}} \arrow[rr] \arrow[d, two heads, "P_{\text{forgchoice}}"'] \arrow[drr, phantom, "\ulcorner"] & & T_2 \times_{T_1}^{12^*,01^*} T_2 \arrow[d, two heads, "P_{21}"] \\
T_1 \arrow[rr, "(11^*,01^*,00^*)"] & & T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1
\end{tikzcd}$$

Lemma 8.27. *The maps* $P_{\text{forgchoice}}, P_{\text{forgcomp}}$ *are Kan fibrations.*

Proof. Exercise 8.16 □

We now have the following immediate observation about P_{forgcomp}.

Lemma 8.28. *The map* $P_{\text{forgcomp}}\colon T_{\text{hoeqcomp}} \to T_{\text{hoeqchoice}}$ *is a trivial fibration of Kan complexes.*

Proof. Exercise 8.17 □

Remark 8.29. In light of the explanations in Intuition 8.22, we can understand a point in the space T_{hoeqcomp} as a homotopy equivalence together with compatible inverses. On the other hand, a point in $T_{\text{hoeqchoice}}$ is a homotopy equivalence together with not necessarily compatible inverses, and the map P_{forgcomp} is a forgetful map that forgets the compatibility. The fact that P_{forgcomp} is a trivial fibration equivalently states that for a morphism $(f, g, h, \sigma_g, \sigma_h)\colon \mathbb{D}^0 \to T_{\text{hoeqchoice}}$, not only is there a lift to T_{hoeqcomp}, which corresponds to a compatibility of the inverses (as proven in Lemma 8.5), but that there is a homotopically unique way to choose such compatibility data (consistent with the philosophy of Section 6).

We can even simplify the pullback diagram to characterize $T_{\text{hoeqcomp}}, T_{\text{hoeqchoice}}$ in a simpler manner.

Lemma 8.30. *In the commutative diagram below*

$$\begin{tikzcd}
T_{\text{hoeqcomp}} \arrow[r, two heads, "P_{\text{forgcomp}}"] \arrow[d] & T_{\text{hoeqchoice}} \arrow[r, two heads, "P_{\text{forgchoice}}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T_1 \arrow[r, two heads, "(1^*,0^*)"] \arrow[d, "(11^*,01^*,00^*)"] & T_0 \times T_0 \arrow[d, "00^* \times 00^*"] \\
T_3 \arrow[r, two heads, "P_{32}"] & T_2 \times_{T_1}^{12^*,01^*} T_2 \arrow[r, two heads, "P_{21}"] & T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1 \arrow[r, two heads, "(\pi_1,\pi_3)"] & T_1 \times T_1
\end{tikzcd}$$

all squares and rectangles are pullback squares.

Proof. The middle square and left-hand rectangle are pullbacks by definition. So, we can apply pullback pasting to deduce that the left-hand square is a pullback. Now, again by pullback pasting, all remaining squares and rectangles will be pullbacks as soon as we show that the right-hand square is a pullback.

However, this follows from Proposition 3.20 and the fact that the following is a pushout diagram

$$\begin{tikzcd}
\Delta^1 \coprod \Delta^1 \arrow[r, "01_0+01_2"] \arrow[d, "00_0+00_1"'] & \Delta^1 \coprod_{\Delta^0}^{1,1} \Delta^1 \coprod_{\Delta^0}^{0,0} \Delta^1 \arrow[d, "11+01+00"] \\
\Delta^0 \coprod \Delta^0 \arrow[r, "1+0"] \arrow[ur, phantom, "\ulcorner"] & \Delta^1
\end{tikzcd} .$$

Here we freely used the Yoneda lemma (Lemma 3.11) to characterize the maps. The subscripts indicate which copy of Δ^1 in $\Delta^1 \coprod_{\Delta^0} \Delta^1 \coprod_{\Delta^0} \Delta^1$ we are mapping into. □

We now want to show that both $T_{\text{hoeqcomp}}, T_{\text{hoeqchoice}}$ are equivalent to T_{hoequiv}. For that we need some additional definitions.

Definition 8.31 (Space of left and right inverses)**.** Let T be a Segal space, let x, y be two objects, and let $f\colon x \to y$ be a morphism. We define the *space of left inverses* $\mathcal{L}\mathrm{Inv}(f)$ and the *space of right inverses* $\mathcal{R}\mathrm{Inv}(f)$ via the following pullback diagrams

$$\begin{tikzcd} \mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & T_2 \arrow[d, two heads, "(02^*,01^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_x,f)"'] & T_1 \times^{0^*,0^*}_{T_0} T_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & T_2 \arrow[d, two heads, "(02^*,12^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_y,f)"'] & T_1 \times^{1^*,1^*}_{T_0} T_1 \end{tikzcd}$$

Intuition 8.32. A point in $\mathcal{L}\mathrm{Inv}(f)$ corresponds to a morphism $h\colon y \to x$ together with a 2-cell from hf to id_x, and a point in $\mathcal{R}\mathrm{Inv}(f)$ corresponds to a morphism $g\colon y \to x$ together with a 2-cell from fg to id_y. Thus, $\mathcal{L}\mathrm{Inv}(f)$ and $\mathcal{R}\mathrm{Inv}(f)$ can be thought of as the spaces of left and right inverses of f respectively.

Lemma 8.33. *Let T be a Segal space, let x, y be two objects, and let $f\colon x \to y$ be a morphism. Then the following is a pullback diagram*

$$\begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \times \mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] & T_{\mathrm{hoeqchoice}} \arrow[d, two heads, "P_{\mathrm{forgchoice}}"] \\ \mathbb{D}^0 \arrow[r, "f"] & T_1 \end{tikzcd}$$

Proof. We can extend the diagram to the following diagram

$$\begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \times \mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] & T_{\mathrm{hoeqchoice}} \arrow[r] \arrow[d, two heads, "P_{\mathrm{forgchoice}}"] & T_2 \times^{12^*,01^*}_{T_1} T_2 \arrow[d, two heads] \\ \mathbb{D}^0 \arrow[r, "f"] & T_1 \arrow[r] & T_1 \times^{1^*,1^*}_{T_0} T_1 \times^{0^*,0^*}_{T_0} T_1 \end{tikzcd},$$

where the right-hand square is a pullback by definition of $T_{\mathrm{hoeqchoice}}$. Hence, by pullback pasting, it suffices to show that the outer rectangle is a pullback. We now have the following diagram

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "(\mathrm{id}_y,f)"] \arrow[d, equal] & T_1 \times^{1^*,1^*}_{T_0} T_1 \arrow[d, "\pi_2"] & T_2 \arrow[l, two heads, "(02^*,12^*)"'] \arrow[d, "12^*"] & \mathcal{R}\mathrm{Inv}(f) \arrow[d] \\ \mathbb{D}^0 \arrow[r, "\{f\}"] & T_1 \arrow[r, equal] & T_1 \arrow[r, squiggly] & \mathbb{D}^0 \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_x,f)"] \arrow[u, equal] & T_1 \times^{0^*,0^*}_{T_0} T_1 \arrow[u, "\pi_2"'] \arrow[d, squiggly] & T_2 \arrow[l, two heads, "(02^*,01^*)"'] \arrow[u, "01^*"'] & \mathcal{L}\mathrm{Inv}(f) \arrow[u] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_y,f,\mathrm{id}_x)"] & T_1 \times^{1^*,1^*}_{T_0} T_1 \times^{0^*,0^*}_{T_0} T_1 & T_2 \times^{12^*,01^*}_{T_1} T_2 \arrow[l, two heads, "P_{21}"'] & \end{tikzcd}.$$

We can directly see that the pullback of the columns gives us the bottom row, which is the pullback we want to compute. On the other hand, by Definition 8.31, the pullbacks of the rows give us the right column, whose pullback is the product $\mathcal{R}\mathrm{Inv}(f) \times \mathcal{L}\mathrm{Inv}(f)$. Finally, by pullback commutativity, these two coincide, giving us the desired result. $\square$

Intuition 8.34. Considering the detailed explanation given in Intuition 8.22, a point in the pullback $T_{\mathrm{hoeqchoice}} \times^{P_{\mathrm{forgchoice}},\{f\}}_{T_1} \mathbb{D}^0$ corresponds to a choice of left and right inverses for the chosen morphism $f\colon x \to y$. Lemma 8.33 is telling us that these choices of data are fully independent not just at the level of vertices, but for the whole simplicial structure, which is consistent with our intuition regarding left and right inverses.

Lemma 8.35. *Let T be a Segal space, let x, y be two objects, and let $f\colon x \to y$ be a morphism. Then the following diagrams are pullback squares*

$$\begin{tikzcd} \mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{map}_T(x,y,x) \arrow[d, two heads, "(02^*,01^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_x, f)"'] & \mathrm{map}_T(x,x) \times \mathrm{map}_T(x,y) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{map}_T(y,x,y) \arrow[d, two heads, "(02^*,12^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_y, f)"'] & \mathrm{map}_T(y,y) \times \mathrm{map}_T(x,y) \end{tikzcd}$$

Proof. We only show the case for $\mathcal{L}\mathrm{Inv}(f)$ and the other case is Exercise 8.18. We have the following diagram

$$\begin{tikzcd} \mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d] & \mathrm{map}_T(x,y,x) \arrow[r] \arrow[d, two heads, "(02^*,01^*)"] & T_2 \arrow[d, two heads, "(02^*,01^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_x, f)"'] & \mathrm{map}_T(x,x) \times \mathrm{map}_T(x,y) \arrow[r] \arrow[d, two heads] & T_1 \times^{0^*,0^*}_{T_0} T_1 \arrow[d, two heads, "(0^*\pi_1, 1^*\pi_2, 1^*\pi_1)"] \\ & \mathbb{D}^0 \arrow[r, "(x,y,x)"] & T_0 \times T_0 \times T_0 \end{tikzcd}.$$

We make the following observations about this diagram. First the right-hand rectangle is a pullback by definition of $\mathrm{map}_T(x,y,x)$. We now move on to show the bottom square is a pullback. We have the following diagram

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "(x,y)"] \arrow[d, equal] & T_0 \times T_0 \arrow[d, "\pi_1"] & & T_1 \arrow[ll, two heads, "(0^*,1^*)"'] \arrow[d, "0^*"] & \mathrm{map}_T(x,y) \arrow[d] \\ \mathbb{D}^0 \arrow[r, "\{x\}"] \arrow[d, equal] & T_0 \arrow[rr, equal] & & T_0 \arrow[r, squiggly] & \mathbb{D}^0 \\ \mathbb{D}^0 \arrow[r, "(x,x)"] & T_0 \times T_0 \arrow[u, "\pi_2"'] \arrow[d, squiggly] & & T_1 \arrow[ll, two heads, "(0^*,1^*)"'] \arrow[u, "0^*"'] & \mathrm{map}_T(x,x) \arrow[u] \\ \mathbb{D}^0 \arrow[r, "(x,y,x)"] & T_0 \times T_0 \times T_0 & & T_1 \times^{0^*,0^*}_{T_0} T_1 \arrow[ll, two heads, "(0^*\pi_1, 1^*\pi_2, 1^*\pi_1)"'] & \end{tikzcd}.$$

We can directly see that the pullback of the columns gives us the bottom row, which is the pullback of the bottom right square. On the other hand, by Definition 8.31, the pullbacks of the rows give us the right column, whose pullback is the product $\mathrm{map}_T(x,x) \times \mathrm{map}_T(x,y)$. Thus, by pullback commutativity, these two coincide, proving the bottom square is indeed a pullback square. Hence, by pullback pasting, the top right square is also a pullback.

Finally, by Definition 8.31, the top rectangle is also a pullback, and so, again by pullback pasting, the left-hand square is also a pullback. Hence we are done. □

Definition 8.36. Let T be a Segal space, let x, y, z be three objects, and let $f\colon x \to y, g\colon y \to z$ be two morphisms. We define $\mathrm{map}_T(x,y,z)_{f/}, \mathrm{map}_T(x,y,z)_{/g}$ as the pullback of the following diagram

$$\begin{tikzcd} \mathrm{map}_T(x,y,z)_{f/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{map}_T(x,y,z) \arrow[d, two heads, "01^*"] \\ \mathbb{D}^0 \arrow[r, "f"'] & \mathrm{map}_T(x,y) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{map}_T(x,y,z)_{/g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{map}_T(x,y,z) \arrow[d, two heads, "12^*"] \\ \mathbb{D}^0 \arrow[r, "g"'] & \mathrm{map}_T(y,z) \end{tikzcd}.$$

We have the following result about these spaces.

Lemma 8.37. *Let T be a Segal space, let x, y, z be three objects, and let $f\colon x \to y, g\colon y \to z$ be two morphisms. Then $02^*\colon \mathrm{map}_T(x,y,z)_{f/} \to \mathrm{map}_T(x,z)$, $02^*\colon \mathrm{map}_T(x,y,z)_{/g} \to \mathrm{map}_T(x,z)$ are Kan fibrations.*

Proof. We prove the case for $\mathrm{map}_T(x,y,z)_{f/}$ and the other case is Exercise 8.19. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{map}_T(x,y,z)_{f/} \arrow[rr] \arrow[d, "02^*"] \arrow[drr, phantom, "\ulcorner", very near start] & & \mathrm{map}_T(x,y,z) \arrow[d, "(02^*{,}01^*)"] \arrow[dd, bend left=60, "01^*"] \\
\mathrm{map}_T(x,z) \arrow[r] \arrow[d] & \mathrm{map}_T(x,z) \times \mathrm{map}_T(x,y) \arrow[r, phantom] & {} \\
\mathbb{D}^0 \arrow[rr, "f"'] & & \mathrm{map}_T(x,y) \arrow[from=2-2, to=3-3, "\pi_2"]
\end{tikzcd} .$$

The bottom square is a pullback by direct computation, and the rectangle is a pullback square, by Definition 8.36, and so the top square is also a pullback square by pullback pasting. As the right-hand vertical map is a Kan fibration, so is the left-hand vertical map (Lemma 4.10), as desired. □

Lemma 8.38. *Let T be a Segal space, let x, y, z be three objects, and let $f\colon x \to y, g\colon y \to z$ be two morphisms. Then the maps $12^*\colon \mathrm{map}_T(x,y,z)_{f/} \to \mathrm{map}_T(y,z)$, $01^*\colon \mathrm{map}_T(x,y,z)_{/g} \to \mathrm{map}_T(x,y)$ are equivalences.*

Proof. We prove the case for $\mathrm{map}_T(x,y,z)_{f/}$ and the other case is Exercise 8.20. We have the following diagram

$$\begin{tikzcd}
\mathrm{map}_T(x,y,z) \arrow[r, two heads] \arrow[d, "(01^*{,}12^*)"', "\simeq"] & \mathrm{map}_T(x,y) \arrow[d, equal] & \mathbb{D}^0 \arrow[l, "\{f\}"'] \arrow[d, equal] \\
\mathrm{map}_T(x,y) \times \mathrm{map}_T(y,z) \arrow[r, two heads, "\pi_1"] & \mathrm{map}_T(x,y) & \mathbb{D}^0 \arrow[l, "\{f\}"']
\end{tikzcd} ,$$

where the vertical map is an equivalence by Proposition 7.22, and the left horizontal maps are Kan fibrations. Hence, by Lemma 4.12 and Remark 4.13, the induced map on pullbacks, namely $\mathrm{map}_T(x,y,z)_{f/} \to \mathrm{map}_T(y,z)$ is also an equivalence, as desired. □

Lemma 8.39. *Let T be a Segal space, let x, y be two objects, and let $f\colon x \to y$ be a morphism. Then the following are pullback diagrams.*

$$\begin{tikzcd}
\mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{map}_T(x,y,x)_{f/} \arrow[d, "02^*"] & \mathcal{R}\mathrm{Inv}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{map}_T(y,x,y)_{/f} \arrow[d, "02^*"] \\
\mathbb{D}^0 \arrow[r, "\mathrm{id}_x"'] & \mathrm{map}_T(x,x) & \mathbb{D}^0 \arrow[r, "\mathrm{id}_y"'] & \mathrm{map}_T(y,y)
\end{tikzcd}$$

Proof. We prove the case for $\mathcal{L}\mathrm{Inv}(f)$ here and the case of $\mathcal{R}\mathrm{Inv}(f)$ is Exercise 8.21. We now have the following diagram

$$\begin{tikzcd}
\mathcal{L}\mathrm{Inv}(f) \arrow[r] \arrow[d] & \mathrm{map}_T(x,y,x)_{f/} \arrow[r] \arrow[d] & \mathrm{map}_T(x,y,x) \arrow[d, "(02^*{,}01^*)"] \\
\mathbb{D}^0 \arrow[r, "\{\mathrm{id}_x\}"] \arrow[rr, bend right, "(\{\mathrm{id}_x\}{,}\{f\})"'] & \mathrm{map}_T(x,x) \arrow[r, "\mathrm{id}\times\{f\}"] & \mathrm{map}_T(x,x) \times \mathrm{map}_T(x,y)
\end{tikzcd} .$$

By Lemma 8.35 the rectangle is a pullback square, and by Definition 8.36 the right square is also a pullback square. Thus, by pullback pasting, the left square is also a pullback square, giving us the desired result. □

We now have the following key observations about $\mathcal{L}\mathrm{Inv}(f)$, $\mathcal{R}\mathrm{Inv}(f)$, when f is a homotopy equivalence.

Proposition 8.40. *Let T be a Segal space, let x, y be two objects, and let $f\colon x \to y$ be a homotopy equivalence. Then $\mathcal{L}\mathrm{Inv}(f)$ and $\mathcal{R}\mathrm{Inv}(f)$ are contractible Kan complexes.*

Proof. We will consider the case $\mathcal{L}\mathrm{Inv}(f)$, and the case for $\mathcal{R}\mathrm{Inv}(f)$ is Exercise 8.22. First we observe that the composition $\mathrm{map}_T(x,y,x)_{f/} \to \mathrm{map}_T(y,x) \xrightarrow{f^*} \mathrm{map}_T(x,x)$ is homotopic to the map $02^*\colon \mathrm{map}_T(x,y,x)_{f/} \to \mathrm{map}_T(x,x)$, as both correspond to pre-composition with f. However, by Lemma 8.38 and Proposition 8.8, the first map is a composition of equivalences and hence an equivalence. This means $02^*\colon \mathrm{map}_T(x,y,x)_{f/} \to \mathrm{map}_T(x,x)$ is also an equivalence. As it is also a Kan fibration (Lemma 8.37), it is in fact a trivial Kan fibration.

By Proposition 6.12, it hence follows that the fiber over $\{\mathrm{id}_x\}$, namely $\mathcal{L}\mathrm{Inv}(f)$ (Lemma 8.39), is contractible. □

Intuition 8.41. It is a very classical result in category theory that if a morphism is an isomorphism, then the choice of (right or left) inverse is unique. Proposition 8.40 is the correct homotopical generalization thereof, proving left and right inverses of equivalences are homotopically unique (Definition 6.6).

Remark 8.42. Notice the condition that f is a homotopy equivalence is crucial in the above result. Indeed, this result even fails in classical category theory, such as the category of sets, where every injection admits many left inverses and every surjection admits many right inverses.

We now have the following elegant result.

Theorem 8.43. *Let T be a Segal space. The map $T_{\mathrm{hoeqchoice}} \to T_1$ factors through $T_{\mathrm{hoequiv}} \hookrightarrow T_1$ and the resulting map $T_{\mathrm{hoeqchoice}} \to T_{\mathrm{hoequiv}}$ is a homotopy equivalence.*

Proof. We first show that the map $T_{\mathrm{hoeqchoice}} \to T_1$ factors through T_{hoequiv}. By definition a point in $T_{\mathrm{hoeqchoice}}$ is a tuple $(f, g, h, \sigma_g, \sigma_h)$, where σ_g, σ_h witness g, h being right and left inverses of f. The map $T_{\mathrm{hoeqchoice}} \to T_1$ takes this tuple to f. Since g, h are inverses of f, f is a homotopy equivalence and so the map factors through T_{hoequiv}.

Now, by Lemma 8.27, $T_{\mathrm{hoeqchoice}} \to T_1$ is a Kan fibration. As $T_{\mathrm{hoequiv}} \to T_1$ is an inclusion of path components, the restricted map $T_{\mathrm{hoeqchoice}} \to T_{\mathrm{hoequiv}}$ remains a Kan fibration (Lemma 4.11). By Proposition 6.12, to prove that $T_{\mathrm{hoeqchoice}} \to T_{\mathrm{hoequiv}}$ is a homotopy equivalence, it suffices to show that for every f in T_{hoequiv}, the fiber over f is contractible. By Lemma 8.33, this fiber is equivalent to $\mathcal{R}\mathrm{Inv}(f) \times \mathcal{L}\mathrm{Inv}(f)$, which is contractible by Proposition 8.40, combined with the fact that f is a homotopy equivalence and the product of contractible Kan complexes is contractible. □

Intuition 8.44. By definition, a point in T_{hoequiv} is a map $f\colon x \to y$ in T that has the *property* of being a homotopy equivalence. On the other hand a point in $T_{\mathrm{hoeqchoice}}$ is a map $f\colon x \to y$ in T and choices of inverses $g, h\colon y \to x$ along with homotopies that witness the inverse properties. The map $P_{\mathrm{forgchoice}}\colon T_{\mathrm{hoeqchoice}} \to T_{\mathrm{hoequiv}}$ forgets the specific chosen inverse and only keeps the map. The proof above basically says that if we know a map is a homotopy equivalence, there is only a contractible way of choosing inverses. Hence, following the philosophy of Section 6, the choice of inverse is homotopically unique.

This observation also relates back to Proposition 8.6, where we showed that we can recover being a homotopy equivalence at the level of homotopy categories. The homotopy category cannot see all these homotopies, as it does not remember the mapping space. Theorem 8.43 confirms that this does not cause any loss of information, as we only need the mere existence of a choice of inverse.

Let us end with the local analogue of the space of homotopy equivalences.

Definition 8.45. Let T be a Segal space. For two objects x, y in T we define the *space of homotopy equivalences*, $\mathrm{hoequiv}_T(x, y)$, as the following pullback

$$\begin{tikzcd} \mathrm{hoequiv}_T(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & T_{\mathrm{hoequiv}} \arrow[d, two heads, "(0^*,1^*)"] \\ \mathbb{D}^0 \arrow[r, "(x,y)"'] & T_0 \times T_0 \end{tikzcd}$$

We have the following result regarding this definition.

Lemma 8.46. *Let T be a Segal space and let x, y be two objects in T. Then the space* $\mathrm{hoequiv}_T(x, y)$ *is isomorphic to the sub-simplicial set of* $\mathrm{map}_T(x, y)$ *generated by the homotopy equivalences (Example 1.32).*

Proof. By Lemma 8.25, the map $T_{\mathrm{hoequiv}} \hookrightarrow T_1$ is an inclusion of path-components, and the map $\mathrm{hoequiv}_T(x, y) \to \mathrm{map}_T(x, y)$ is given as the pullback of the following diagram

$$\begin{tikzcd} T_{\mathrm{hoequiv}} \arrow[r, two heads, "(0^*,1^*)"] \arrow[d, hook] & T_0 \times T_0 \arrow[d, equal] & \mathbb{D}^0 \arrow[l, "(x,y)"'] \arrow[d, equal] \\ T_1 \arrow[r, two heads, "(0^*,1^*)"] & T_0 \times T_0 & \mathbb{D}^0 \arrow[l, "(x,y)"'] \end{tikzcd} .$$

This means it again satisfies the pullback condition defining inclusion of path-components (Definition 4.22). □

Exercises.

Exercise 8.1. Let T be a Segal space and let $f\colon x \to y$ be a morphism. Explain why a right inverse of f is naturally encoded by a point of a suitable pullback involving

$$\mathrm{map}_T(y, x, y),$$

while a left inverse of f is naturally encoded by a point of a suitable pullback involving

$$\mathrm{map}_T(x, y, x).$$

Exercise 8.2. Let T be a Segal space. Unpack the definition of the space of homotopy equivalences

$$T_{\mathrm{hoequiv}} \subseteq T_1.$$

Explain why it is defined as the sub-simplicial set generated by certain vertices, rather than merely as a subset of $(T_1)_0$.

Exercise 8.3. Let T be a Segal space. Describe explicitly what a vertex of

$$T_{\mathrm{hoeqchoice}}$$

represents. Then describe explicitly what extra data a vertex of

$$T_{\mathrm{hoeqcomp}}$$

represents.

Exercise 8.4. Prove the forward implication of Proposition 8.6: if

$$f\colon x \to y$$

is a homotopy equivalence in a Segal space T, then $[f]$ is an isomorphism in $\mathrm{Ho}(T)$.

Exercise 8.5. Prove pre-composition case of Proposition 8.8, that is, prove that f is a homotopy equivalence if and only if for every object z, the map

$$f^*\colon \mathrm{map}_T(y, z) \to \mathrm{map}_T(x, z)$$

is a homotopy equivalence.

Exercise 8.6. Prove Lemma 8.10: A Segal space T is a Segal space groupoid if and only if $\mathrm{Ho}(T)$ is a groupoid.

Exercise 8.7. Prove Lemma 8.12: A functor of Segal spaces $F\colon T \to U$ is essentially surjective if and only if $\mathrm{Ho}(F)\colon \mathrm{Ho}(T) \to \mathrm{Ho}(U)$ is essentially surjective.

Exercise 8.8. Prove Lemma 8.14: There is an isomorphism of simplicial sets

$$\mathrm{Map}(Z(3), T) \cong T_1 \times_{T_0}^{1^*,1^*} T_1 \times_{T_0}^{0^*,0^*} T_1,$$

using Proposition 3.20.

Exercise 8.9. Complete the first part of the proof of Lemma 8.15: show that the diagram

$$\begin{tikzcd}
 & \Delta^0 \arrow[dl, "1"'] \arrow[dr, "1"] & & \Delta^0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\
\Delta^1 \arrow[drr, "02_0"'] & & \Delta^1 \arrow[d, "12_0"] & & \Delta^1 \arrow[dll, "02_1"] \\
 & & \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 & &
\end{tikzcd},$$

is commutative.

Exercise 8.10. Complete the second part of the proof of Lemma 8.15: use similar methods to construct a map of simplicial spaces

$$012 + 123\colon \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 \to \Delta^3.$$

Exercise 8.11. Prove Lemma 8.16 using the methods in Lemma 8.15. That is, construct the map

$$\Delta^1 \coprod_{\Delta^0}^{1,0} \Delta^1 \coprod_{\Delta^0}^{1,0} \Delta^1 \to \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2$$

given by

$$01_0 + 12_0 + 12_1$$

and show that it induces the Kan fibration

$$C_{21}\colon T_2 \times_{T_1}^{12^*,01^*} T_2 \twoheadrightarrow T_1 \times_{T_0}^{1^*,0^*} T_1 \times_{T_0}^{1^*,0^*} T_1.$$

Exercise 8.12. Complete the first step of the proof of Lemma 8.17: Prove the map

$$(01+12)+(12+23)\colon \mathrm{Sp}^2 \coprod_{\Delta^1}^{12,01} \mathrm{Sp}^2 \to \mathrm{Sp}^3$$

is an isomorphism using arguments analogous to Lemma 1.49.

Exercise 8.13. Complete the next step of the proof of Lemma 8.17: Prove the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sp}^2 \coprod_{\Delta^1}^{12,01} \mathrm{Sp}^2 \arrow[rr, "(01+12)+(12+23)", "\cong"'] \arrow[dr, "i_2+i_2"'] & & \mathrm{Sp}^3 \arrow[dl, "01_0+01_1+12_1"] \\
 & \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 &
\end{tikzcd}.$$

Exercise 8.14. Use the $n = 3$ case of Lemma 1.49 to prove the following diagram in the proof of Lemma 8.18 commutes:

$$\begin{tikzcd} & \mathrm{Sp}^3 \arrow[dl, "01_0+12_0+12_1"'] \arrow[dr, "01+12+23"] & \\ \Delta^2 \coprod_{\Delta^1}^{12,01} \Delta^2 \arrow[rr, "012+123"'] & & \Delta^3 \end{tikzcd},$$

Exercise 8.15. Use the simplicial identities to complete the proof of Lemma 8.19 and prove the following diagram commutes:

$$\begin{tikzcd} & & T_1 \arrow[dll, "11^*"'] \arrow[d, "01^*"] \arrow[drr, "00^*"] & & \\ T_1 \arrow[dr, "1^*"] & & T_1 \arrow[dl, "1^*"'] \arrow[dr, "0^*"] & & T_1 \arrow[dl, "0^*"] \\ & T_0 & & T_0 & \end{tikzcd},$$

Exercise 8.16. Use Lemma 8.15 to prove Lemma 8.27: Show the maps

$$P_{\mathrm{forgchoice}} \colon T_{\mathrm{hoeqchoice}} \to T_1 \qquad \text{and} \qquad P_{\mathrm{forgcomp}} \colon T_{\mathrm{hoeqcomp}} \to T_{\mathrm{hoeqchoice}}$$

are Kan fibrations.

Exercise 8.17. Use Lemma 6.13 to prove Lemma 8.28: $P_{\mathrm{forgcomp}} \colon T_{\mathrm{hoeqcomp}} \to T_{\mathrm{hoeqchoice}}$ is a trivial fibration of Kan complexes.

Exercise 8.18. Prove Lemma 8.35 for $\mathcal{R}\mathrm{Inv}(f)$. That is, show the following is a pullback diagram:

$$\begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{map}_T(y,x,y) \arrow[d, two heads, "(02^*,12^*)"] \\ \mathbb{D}^0 \arrow[r, "(\mathrm{id}_y, f)"'] & \mathrm{map}_T(y,y) \times \mathrm{map}_T(x,y) \end{tikzcd}$$

Exercise 8.19. Prove the second part of Lemma 8.37: the map $02^* \colon \mathrm{map}_T(x,y,z)_{/g} \to \mathrm{map}_T(x,z)$ is a Kan fibration.

Exercise 8.20. Prove the second part of Lemma 8.38, the map $01^* \colon \mathrm{map}_T(x,y,z)_{/g} \to \mathrm{map}_T(x,y)$ is an equivalence.

Exercise 8.21. Prove the second part of Lemma 8.39: prove the following is a pullback diagram:

$$\begin{tikzcd} \mathcal{R}\mathrm{Inv}(f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{map}_T(y,x,y)_{/f} \arrow[d, two heads, "02^*"] \\ \mathbb{D}^0 \arrow[r, "\mathrm{id}_y"'] & \mathrm{map}_T(y,y) \end{tikzcd}.$$

Exercise 8.22. Prove the second part of Proposition 8.40: for every homotopy equivalence f, $\mathcal{R}\mathrm{Inv}(f)$ is a contractible Kan complex.

Exercise 8.23. Let $\mathcal{C}$ be an ordinary category and let

$$T = \mathrm{disc}(N\mathcal{C}).$$

Show that a morphism in T is a homotopy equivalence if and only if the corresponding morphism in $\mathcal{C}$ is an isomorphism.

Exercise 8.24. Let $\mathcal{C}$ be an ordinary category and let

$$T = \mathrm{disc}(N\mathcal{C}).$$

Compute

$$T_{\text{hoequiv}}$$

explicitly as a constant simplicial set. What is its set of vertices?

Exercise 8.25. Let $T = \Delta^2 = \operatorname{disc}(\mathbb{D}^2)$. Compute all morphisms in T. Which of them are homotopy equivalences?

Exercise 8.26. Let G be a group, regarded as a one-object category, and let

$$T = \operatorname{disc}(NG).$$

Compute

$$T_{\text{hoequiv}}.$$

Then compute the local space

$$\operatorname{hoequiv}_T(*, *).$$

Exercise 8.27. Let M be a monoid, regarded as a one-object category, and let

$$T = \operatorname{disc}(NM).$$

Show that the homotopy equivalences in T are precisely the units of M.

Exercise 8.28. Let $T = \Delta^1$ and let

$$f = \mathrm{id}_0.$$

Compute

$$\mathcal{L}\mathrm{Inv}(f), \qquad \mathcal{R}\mathrm{Inv}(f), \qquad \mathcal{R}\mathrm{Inv}(f) \times \mathcal{L}\mathrm{Inv}(f).$$

Exercise 8.29. Let $T = \operatorname{disc}(N\mathcal{C})$ for an ordinary category $\mathcal{C}$, and let

$$f \colon x \to y$$

be an isomorphism in $\mathcal{C}$. Compute

$$\mathcal{L}\mathrm{Inv}(f) \qquad \text{and} \qquad \mathcal{R}\mathrm{Inv}(f).$$

Explain why they are constant simplicial sets on singleton sets.

Exercise 8.30. Compute the set of $Z(3)_{10}$. Then compute its non-degenerate 1-simplices. Draw the resulting directed graph.

Exercise 8.31. Compute $\operatorname{Map}(Z(3), \Delta^2)$ on vertices.

Exercise 8.32. Let $T = \operatorname{disc}(NG)$ for a group G, and let $f \in G$. Compute explicitly the fiber of

$$T_{\text{hoeqchoice}} \to T_{\text{hoequiv}}$$

over f. Identify the unique inverse appearing in this fiber.

Exercise 8.33. Let T be a Segal space and let $f \colon x \to y$ be a morphism. Suppose

$$\mathcal{L}\mathrm{Inv}(f) \qquad \text{and} \qquad \mathcal{R}\mathrm{Inv}(f)$$

are both non-empty. Show that f is a homotopy equivalence.

Exercise 8.34. Let T be a Segal space groupoid. Compute

$$T_{\text{hoequiv}}.$$

What does the inclusion

$$T_{\text{hoequiv}} \hookrightarrow T_1$$

become in this case?

Exercise 8.35. Let T be a Segal space. Suppose

$$T_{\text{hoequiv}} = T_1$$

as simplicial sets. Show that T is a Segal space groupoid.

Exercise 8.36. Let T be a Segal space and let x be an object. Show that

$$\mathrm{id}_x$$

is a homotopy equivalence. Then compute a point of

$$\mathcal{L}\mathrm{Inv}(\mathrm{id}_x) \qquad \text{and} \qquad \mathcal{R}\mathrm{Inv}(\mathrm{id}_x).$$

Exercise 8.37. Let T be a Segal space and let

$$f\colon x \to y, \qquad g\colon y \to z$$

be homotopy equivalences. Prove that any chosen composite

$$g \circ f\colon x \to z$$

is again a homotopy equivalence.

Exercise 8.38. Let $F\colon T \to U$ be a functor of Segal spaces. Show that if

$$f\colon x \to y$$

is a homotopy equivalence in T, then

$$Ff\colon Fx \to Fy$$

is a homotopy equivalence in U.

9. Complete Segal Spaces

Up until this point we defined Segal spaces and developed their category theory. The comparison with the case of classical categories (Corollary 2.18) suggests that Segal spaces are a good model for higher categories. However, in this section we will see that in the higher categorical setting we need one additional property, namely completeness. This work is fundamentally based on [Rez01, Section 3, Section 6].

9.1. Examples of Segal Spaces.

Before we proceed, we want to understand two examples of interest.

Example 9.1 (Nerve as a Segal space)**.** Let $\mathcal{C}$ be a category. Then $\mathrm{disc}(N\mathcal{C})$ is a discrete simplicial space, and hence Reedy fibrant (Proposition 5.8). Moreover, it satisfies the Segal condition (Proposition 2.13). Thus $\mathrm{disc}(N\mathcal{C})$ is a Segal space.

Fortunately, the definitions we are used to from category theory perfectly match up with the ones for Segal space theory. In particular, an object in the Segal space $\mathrm{disc}(N\mathcal{C})$ is just an object in the category $\mathcal{C}$. The same is true for morphisms.

However, as $\mathrm{disc}(N\mathcal{C})_1$ is just a set, the mapping space is actually just a set as well. This in particular implies that composition is well-defined not just up to homotopy. In fact for any collection of objects $x_0, ..., x_n$ in $\mathrm{disc}(N\mathcal{C})$, the space $\mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_0, ..., x_n)$ is in bijection with $\mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_0, x_1) \times ... \times \mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_{n-1}, x_n)$. Thus the pullback

$$\begin{tikzcd}
\mathrm{Comp}_{\mathrm{disc}(N\mathcal{C})}(f,g) \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_0, x_1, x_2) \arrow[d, "(01^*,12^*)"] \\
\mathbb{D}^0 \arrow[r, "(f,g)"'] & \mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_0, x_1) \times \mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_1, x_2)
\end{tikzcd}$$

is not just contractible, but actually just a point.

In addition to all of these, as $\mathrm{map}_{\mathrm{disc}(N\mathcal{C})}(x_0, x_1)$ is just a set, two maps are homotopic if and only if they are equal to each other. This implies that a map is a homotopy equivalence if and only if it is an isomorphism. In particular, the homotopy category of the Segal space, $\mathrm{Ho}(\mathrm{disc}(N\mathcal{C}))$ is exactly $\mathcal{C}$, as the two categories have the same objects and the same morphisms.

Example 9.2 (Spaces as Segal spaces)**.** Let K be a space (Kan complex). Our first guess for the corresponding Segal space might be to take the simplicial space $\mathrm{const}(K)$. However, it is not Reedy fibrant (Example 5.11). Fortunately, there is an equivalent simplicial space that is Reedy fibrant. Namely, let $cK = \mathrm{Map}(\mathbb{D}^\bullet, K)$ be the simplicial space defined as

$$K \underset{t}{\overset{s}{\leftleftarrows\!\!\!\rightarrow}} \mathrm{Map}(\mathbb{D}^1, K) \ \substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow} \ \mathrm{Map}(\mathbb{D}^2, K) \ \substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow} \ \cdots$$

where the boundary maps are induced by the maps between the simplices. Alternatively we can see that $cK_{k,l} = \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\mathbb{D}^k \times \mathbb{D}^l, K)$. We shall take for granted that cK is indeed a Segal space, and refer the reader to [GJ09, Corollary 4.3, Corollary 4.6].

What does the category theory of a Segal space look like in this case? An object in this Segal space cK is a point in K. A morphism is now a point in the space $\mathrm{Map}(\mathbb{D}^1, K)$ and so is just a path $\mathbb{D}^1 \to K$. Composition of morphisms corresponds to concatenation of paths in the space. As concatenation of paths is not unique and only well-defined up to homotopy, here we do need the contractibility condition for composition (Proposition 7.24).

More generally (using Definition 4.18), for two given points x, y,

$$\mathrm{map}_{cK}(x, y) = \mathbb{D}^0 \times_{(K\times K)} \mathrm{Map}(\mathbb{D}^1, K) = \mathcal{P}\mathrm{ath}_K(x, y).$$

So, two morphisms are homotopic if they are homotopic as paths in K. In particular, every morphism is a homotopy equivalence, as every path $\mathbb{D}^1 \to K$ admits an inverse path (explicitly constructed in Lemma 2.41 to show homotopy is symmetric). Thus cK is an example of a Segal space groupoid (Definition 8.9).

Notice that the homotopy category of this Segal space is the category which has objects the points x in K and has morphisms homotopy classes of paths in K, $\pi_0(\mathcal{P}\mathrm{ath}_K(x, y))$. This category is commonly called the *fundamental groupoid* of K and is denoted by $\Pi_1(K)$ [tD08, Section 2.5].

9.2. **Why are Segal Spaces not Enough?** By definition, a Segal space has both a category theory and a homotopy theory. However, there is no condition that guarantees that these two theories are compatible with each other. This causes several problems, which we will illustrate in this section. For this first example, let us recall the groupoid $I(1)$. Then, following Example 9.1, $\mathrm{disc}NI(1)$ is a Segal space.

Notation 9.3. We denote the Segal space $\mathrm{disc}NI(1)$ by $E(1)$.

Example 9.4. Recall that the functor $0\colon [0] \to I(1)$ is an equivalence of categories. However, the map of Segal spaces $\mathrm{disc}N[0] = \Delta^0 \to E(1)$ is not an equivalence of Segal spaces. Indeed, by Lemma 7.41, if it were the case, then

$$\{0\}\colon (\mathrm{disc}N[0])_0 \to (E(1))_0 = \{0, 1\}$$

would be a homotopy equivalence of spaces, which is evidently not the case.

Intuition 9.5. What essentially happened here is that the category theory has an underlying homotopy theory of groupoids ($I(1)$ is a groupoid), which is completely ignored and thus missed by the Segal space.

Example 9.6. Let us go back to $E(1)$ once more. It is a discrete Segal space, with two objects denoted $0, 1$. Moreover, it has two non-identity morphisms, $01, 10$, which are inverses of each other. Thus the two objects are equivalent to each other in the sense that there is a homotopy equivalence between them. However, they are not equivalent in the space $E(1)_0 = \{0, 1\}$, as there is no path between them.

Intuition 9.7. Here we see a clear mismatch between homotopy theory and category theory. Categorically the two points are equivalent, but homotopically they are not.

For this next example, we recall equivalent characterizations of equivalence of categories.

Example 9.8. We would expect that a map of Segal spaces $f: T \to U$ is an equivalence of Segal spaces if it is fully faithful and essentially surjective. However, this already fails for the simple example $\mathrm{disc}N[0] \to \mathrm{disc}NI(1)$.

Intuition 9.9. As in the previous example, the problem is that x and y are equivalent in the Segal space $E(1)$, but as points in the space $E(1)_0$ they are not homotopic.

Example 9.10. In Definition 8.9 we defined a Segal groupoid as a Segal space in which every morphism is a homotopy equivalence. In Example 9.2 we discussed how every space K gives us a Segal groupoid cK. However, the opposite is not true, meaning it is not the case that for every Segal groupoid W, there exists a space K and an equivalence $W \simeq cK$. Indeed, the Segal space $E(1)$ is a Segal groupoid, but not equivalent to a space.

Intuition 9.11. This example is contrary to our understanding of higher category theory. Intuitively, a higher category has homotopical data and categorical data. However, in a groupoid every morphism is invertible, which means it does not contain any non-trivial categorical data. Therefore, our notion of groupoid should really correspond to just a space. The idea we just explained is commonly called the *homotopy hypothesis* and is one of the guiding ideas in the realm of higher category theory, going back to ideas of Grothendieck in the 80s (republished recently in [Gro22]).

Seeing these examples, we realize that we need to impose one additional condition to make sure the homotopy and category theory suitably coincide.

9.3. **Defining Complete Segal Spaces.** In Subsection 9.2, we saw that Segal spaces fall short of our expectations for a good model of higher categories. In particular, we saw that the homotopy theory and category theory of a Segal space are completely disconnected from each other, and we need to impose some compatibility condition. This will result in complete Segal spaces, which is the focus of this section.

Lemma 9.12. *Let T be a Segal space and let x be an object in T. Then the identity morphism id_x is a homotopy equivalence.*

Proof. Exercise 9.14 □

Remark 9.13. Lemma 9.12 implies that the image of $00^*: T_0 \to T_1$ is actually contained in T_{hoequiv}, meaning there is a factorization $T_0 \xrightarrow{00^*} T_{\mathrm{hoequiv}} \hookrightarrow T_1$.

This map $00^*: T_0 \to T_{\mathrm{hoequiv}}$ is generally not an equivalence, as the following example shows.

Example 9.14. In the Segal space $E(1)$, the space of objects is the set $\{0,1\}$, but the space of morphisms is the set with four elements $\{00, 01, 10, 11\}$, all of which are equivalences. Thus the map

$$00^*: \{0,1\} \to \{00, 01, 10, 11\}$$

is not an equivalence, as it is not surjective.

We saw in several examples of Subsection 9.2 that the lack of paths between equivalent objects is the source of all the problems. We therefore introduce the next definition to address this issue.

Definition 9.15 (Complete Segal space)**.** A *complete Segal space* is a Segal space W for which the map

$$00^*: W_0 \to W_{\mathrm{hoequiv}}$$

defined in Remark 9.13 is an equivalence.

We now want to characterize the completeness condition in several equivalent ways. Before we can do that, we need to make several technical observations and constructions.

Remark 9.16. Given a Kan fibration $p\colon Y \twoheadrightarrow X$ and a Kan complex K, every commutative square of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d, "00^*"'] & Y \arrow[d, "p"] \\ \mathrm{Map}(\mathbb{D}^1, K) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

admits a lift. For a proof see [GJ09, First paragraph of Section I.9, Theorem 11.3].

Lemma 9.17. *Let W be a Segal space. Then the map*

$$W_{\mathrm{hoequiv}} \to W_0 \times W_0$$

is a Kan fibration.

Proof. Exercise 9.15 □

Construction 9.18. Let W be a Segal space. Then following Remark 9.16, the following diagram admits a lift

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & W_{\mathrm{hoequiv}} \arrow[d, two heads] \\ \mathrm{Map}(\mathbb{D}^1, W_0) \arrow[r] \arrow[ur, dashed, "P"] & W_0 \times W_0 \end{tikzcd},$$

that we denote $P\colon \mathrm{Map}(\mathbb{D}^1, W_0) \to W_{\mathrm{hoequiv}}$.

Intuition 9.19. Intuitively, this map takes a path $\gamma\colon \mathbb{D}^1 \to W_0$ from x to y and produces a homotopy equivalence from x to y.

Definition 9.20. Let W be a Segal space and let x, y be two objects in W. Let

$$P_{x,y}\colon \mathcal{P}\mathrm{ath}_{W_0}(x,y) \to \mathrm{hoequiv}_W(x,y)$$

be given by the pullback of the following diagram

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "{(x,y)}"] \arrow[d, equal] & W_0 \times W_0 \arrow[d, equal] & \mathrm{Map}(\mathbb{D}^1, W_0) \arrow[l, two heads] \arrow[d, "P"] \\ \mathbb{D}^0 \arrow[r, "{(x,y)}"] & W_0 \times W_0 & W_{\mathrm{hoequiv}} \arrow[l, two heads] \end{tikzcd}$$

Theorem 9.21 ([Rez01, Theorem 6.2])**.** *Let W be a Segal space. Then there is a homotopy equivalence $W_{\mathrm{hoequiv}} \simeq \mathrm{Map}(E(1), W)$.*

Proposition 9.22. *Let W be a Segal space. The following are equivalent.*

(1) *W is a complete Segal space.*
(2) *Given the commutative diagram below,*

$$\begin{tikzcd} W_0 \arrow[rr, "0000^*"] \arrow[d, "00^*"'] & & W_3 \arrow[d, two heads, "{(02^*,12^*,13^*)}"] \\ W_1 \arrow[r, "{(11^*,01^*,00^*)}"] & W_1 \times^{1^*,1^*}_{W_0} W_1 \times^{0^*,0^*}_{W_0} W_1 & \end{tikzcd},$$

the induced map $W_0 \to W_{\mathrm{hoeqcomp}}$ (Definition 8.26) is a homotopy equivalence.
(3) *Given the commutative diagram below,*

$$\begin{tikzcd} W_0 \arrow[r, "{(000^*,000^*)}"] \arrow[d, "00^*"'] & W_2 \times^{12^*,01^*}_{W_1} W_2 \arrow[d, two heads, "P_{21}"] \\ W_1 \arrow[r, "{(11^*,01^*,00^*)}"] & W_1 \times^{1^*,1^*}_{W_0} W_1 \times^{0^*,0^*}_{W_0} W_1 \end{tikzcd},$$

the induced map $W_0 \to W_{\mathrm{hoeqchoice}}$ (Definition 8.26) is a homotopy equivalence.

(4) *The map of spaces*
$$0^*\colon \mathrm{Map}(E(1), W) \to \mathrm{Map}(\Delta^0, W),$$
induced by the inclusion $0\colon \Delta^0 \to E(1)$*, is a weak equivalence.*

(5) *For any two objects* x, y *the natural map (Definition 9.20)*
$$P_{x,y}\colon \mathcal{P}\mathrm{ath}_{W_0}(x,y) \to \mathrm{hoequiv}_W(x,y)$$
is an equivalence of spaces.

Proof. We consider various equivalent conditions.

($1 \iff 2 \iff 3$) We have the following diagram

$$\begin{tikzcd}
 & & W_0 \arrow[dll] \arrow[d] \arrow[drr] & & \\
W_{\mathrm{hoeqcomp}} \arrow[rr, two heads, "\simeq", "P_{\mathrm{forgcomp}}"'] & & W_{\mathrm{hoeqchoice}} \arrow[rr, two heads, "\simeq", "P_{\mathrm{forgchoice}}"'] & & W_{\mathrm{hoequiv}}
\end{tikzcd}$$

By Theorem 8.43 and Lemma 8.28, the horizontal maps are trivial Kan fibrations. Thus, the equivalence of (1), (2) and (3) follows from applying 2-out-of-3 (Remark 2.46).

($1 \iff 4$) By Theorem 9.21, there is an equivalence $\mathrm{Map}(E(1), W) \simeq W_{\mathrm{hoequiv}}$. Thus, by 2-out-of-3 (Remark 2.46), the map $\mathrm{Map}(E(1), W) \to \mathrm{Map}(\Delta^0, W)$ is an equivalence if and only if $W_{\mathrm{hoequiv}} \to W_0$ is an equivalence, which is exactly the completeness condition.

($1 \iff 5$) By Construction 9.18, the map $W_0 \to W_{\mathrm{hoequiv}}$ factors as $W_0 \to \mathrm{Map}(\mathbb{D}^1, W_0) \to W_{\mathrm{hoequiv}}$. Thus, by 2-out-of-3 (Remark 2.46), the map $W_0 \to W_{\mathrm{hoequiv}}$ is an equivalence if and only if the map $P\colon \mathrm{Map}(\mathbb{D}^1, W_0) \to W_{\mathrm{hoequiv}}$ is an equivalence.

Now, again by Construction 9.18, we have the following diagram

$$\begin{tikzcd}
\mathrm{Map}(\mathbb{D}^1, W_0) \arrow[rr] \arrow[dr, two heads] & & W_{\mathrm{hoequiv}} \arrow[dl, two heads] \\
 & W_0 \times W_0 &
\end{tikzcd}.$$

Here, the left diagonal map is a Kan fibration, by Proposition 4.15, and the right diagonal map is a Kan fibration by Lemma 4.9, as it is the composition of the Kan fibration $W_{\mathrm{hoequiv}} \to W_1$ (Lemma 8.25) and $W_1 \to W_0 \times W_0$ (Example 5.4). Thus, by Lemma 4.14, the top map is a homotopy equivalence if and only if, for every pair of points (x, y), the map between fibers is an equivalence. However, by Definition 9.20, this is precisely the map
$$P_{x,y}\colon \mathcal{P}\mathrm{ath}_{W_0}(x,y) \to \mathrm{hoequiv}_W(x,y).$$
□

Intuition 9.23. The completeness condition exactly addresses the problems we raised in Subsection 9.2. By imposing the additional condition that every equivalence in the Segal space W can be represented by a path in W_0, we are ensuring that the homotopy theory of the space W_0 corresponds to equivalences in W_1.

We can now see how the completeness condition addresses the problems we raised in Subsection 9.2. First we have the following result, the proof of which can be found in [Rez01].

Theorem 9.24 ([Rez01, Proposition 7.6])**.** *Let* $F\colon W \to V$ *be a functor of complete Segal spaces. The following are equivalent:*

(1) F *is an equivalence.*
(2) F *is fully faithful and essentially surjective.*

We now have a similar result regarding the homotopy hypothesis (Intuition 9.11).

Definition 9.25 (Complete Segal space groupoid)**.** A *complete Segal space groupoid* is a complete Segal space W such that every morphism in W is a homotopy equivalence.

In Example 9.10, we saw that not every Segal space groupoid is equivalent to a Segal space cK, for some space K. We now want to see how the completeness condition rectifies this problem.

Example 9.26. Let K be a space. Then the Segal space cK is a complete Segal space groupoid. Indeed, we already saw in Example 9.2 that cK is a Segal space groupoid. Moreover, we saw in the aforementioned example that $\mathrm{map}_{cK}(x,y) \simeq \mathcal{P}\mathrm{ath}_K(x,y)$ for any x, y in K. Hence in this case the map $P_{x,y}\colon \mathcal{P}\mathrm{ath}_K(x,y) \to \mathrm{hoequiv}_{cK}(x,y)$ is the identity, and so cK is complete, by Proposition 9.22.

On the other hand, we have the following major result, which confirms the homotopy hypothesis we discussed in Intuition 9.11.

Proposition 9.27. *A complete Segal space W is a complete Segal space groupoid if and only if W is equivalent to cW_0. Hence complete Segal spaces satisfy the homotopy hypothesis.*

Proof. Exercise 9.19 □

9.4. **Rezk Nerves as Complete Segal Spaces.** Our final goal is to discuss how we can build a complete Segal space out of a category. We have seen already that $\mathrm{disc}N\mathcal{C}$ is a Segal space (Example 9.1), but is not necessarily complete (Example 9.14). Before we can rectify the situation let us better comprehend the problem at hand.

Remark 9.28. We saw in Example 9.1 that objects and morphisms in $\mathrm{disc}N\mathcal{C}$ are the objects and morphisms of $\mathcal{C}$ and a morphism in the Segal space $\mathrm{disc}N\mathcal{C}$ is a homotopy equivalence if and only if it is an isomorphism in $\mathcal{C}$. Hence, for two objects x, y in $\mathcal{C}$, $\mathrm{hoequiv}_{\mathrm{disc}N\mathcal{C}}(x,y) \cong \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y)$. Here we used the fact that the morphisms in the underlying groupoid $\mathcal{C}^{\simeq}$ are precisely the isomorphisms in $\mathcal{C}$.

Given this explanation and the content of Proposition 9.22, the completeness condition means $\mathcal{P}\mathrm{ath}_{(\mathrm{disc}N\mathcal{C})_0}(x,y) \simeq \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y)$. However, $(\mathrm{disc}N\mathcal{C})_0 = \mathrm{Obj}_{\mathcal{C}}$ is just the set of objects of $\mathcal{C}$, and so

$$\mathcal{P}\mathrm{ath}_{(\mathrm{disc}N\mathcal{C})_0}(x,y) = \begin{cases} \varnothing & x \neq y \\ \{\mathrm{id}_x\} & x = y \end{cases}.$$

What we need is a suitably adapted version of the nerve construction that not only adds objects and compositions in the horizontal directions, but also isomorphisms in the vertical direction. This leads to the following construction, which is due to Rezk [Rez01].

For this definition, we use the evident inclusion $I\colon \mathbb{\Delta} \to \mathcal{C}\mathrm{at}$ (Definition 1.1), the opposite category $(-)^{\mathrm{op}}$, the functor category, the core of a category, and the nerve of a category (Definition 2.1).

Definition 9.29 (Rezk nerve)**.** Let $\mathcal{C}$ be a category. We define the *Rezk nerve* of $\mathcal{C}$ as the simplicial space $\mathcal{N}\mathcal{C}$ given by the following composition

$$\mathbb{\Delta}^{\mathrm{op}} \xrightarrow{I^{\mathrm{op}}} \mathcal{C}\mathrm{at}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{C})} \mathcal{C}\mathrm{at} \xrightarrow{(-)^{\simeq}} \mathcal{C}\mathrm{at} \xrightarrow{N} s\mathcal{S}\mathrm{et}.$$

Remark 9.30. Unlike most other definitions in this section, the Rezk nerve is originally called the *classification diagram* in [Rez01]. Given the non-descriptive nature of this name, and the recent prevalence of the terminology "Rezk nerve" in the literature, we have opted to use the modern name in this text.

Before we proceed we want to understand this definition more explicitly.

Remark 9.31. By construction, $\mathcal{N}\mathcal{C}_n = N(\mathrm{Fun}([n],\mathcal{C})^{\simeq})$, meaning $\mathcal{N}\mathcal{C}_{nl} = N(\mathrm{Fun}([n],\mathcal{C})^{\simeq})_l$.

Let us see an even more explicit way to characterize the Rezk nerve of a category.

Intuition 9.32. Following Remark 9.31, $\mathcal{N}\mathcal{C}_0 = N(\mathrm{Fun}([0],\mathcal{C})^{\simeq}) = N(\mathcal{C}^{\simeq})$. Thus two objects are connected via a path in $\mathcal{N}\mathcal{C}_0$ precisely when the objects are isomorphic in $\mathcal{C}$. This is already an improvement over what we saw in Remark 9.28.

We denote the groupoid with $l+1$ objects and a unique isomorphism between any two objects by $I(l)$. In particular, $I(0) = [0]$ and $I(1)$ is the groupoid we first saw in Lemma 4.2.

Lemma 9.33. *Let $\mathcal{C}$ be a category. Then we have the following bijection*

$$\mathcal{N}(\mathcal{C})_{nl} \cong \mathrm{Hom}_{\mathcal{C}at}([n] \times I(l), \mathcal{C}).$$

Proof. By Remark 9.31, $\mathcal{N}(\mathcal{C})_{nl} = N(\mathrm{Fun}([n],\mathcal{C})^{\simeq})_l \cong \mathrm{Hom}_{\mathcal{C}at}([l], \mathrm{Fun}([n],\mathcal{C})^{\simeq})$. However, every morphism in the core is actually an isomorphism, so this canonically coincides with functors $I(l) \to \mathrm{Fun}([n],\mathcal{C})^{\simeq}$. Finally, this is naturally in bijection with the set of functors $[n] \times I(l) \to \mathcal{C}$. □

Intuition 9.34. With Lemma 9.33 at hand, we can depict $\mathcal{N}\mathcal{C}$ very explicitly. Here, we use the following notation.

(1) $\bullet$: objects
(2) $\longrightarrow$: morphisms
(3) $\xrightarrow{\sim}$: isomorphisms

With this notation the Rezk nerve has the form of the following simplicial space:

$$
\begin{array}{ccccccc}
\{\bullet\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \{\bullet \to \bullet\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \{\bullet \to \bullet \to \bullet\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \\
\left\{\begin{matrix}\bullet \\ \downarrow\wr \\ \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \left\{\begin{matrix}\bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \left\{\begin{matrix}\bullet & \to & \bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet & \to & \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow & & \\
\left\{\begin{matrix}\bullet \\ \downarrow\wr \\ \bullet \\ \downarrow\wr \\ \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} & \left\{\begin{matrix}\bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \left\{\begin{matrix}\bullet & \to & \bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet & \to & \bullet \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \bullet & \to & \bullet & \to & \bullet\end{matrix}\right\} & \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} & \cdots \\
\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

As we can see, the vertical direction (the homotopical direction) focuses on the isomorphisms and so the homotopical data of a category, whereas the horizontal direction (the categorical direction) focuses on arbitrary morphisms and its category theory.

Remark 9.35. The Rezk nerve is a special case of a more general construction, known as the *relative nerve* or *classification diagram*. See [Rez01, Section 3] for more details.

Having improved our definition of a nerve, we now proceed to show it is a complete Segal space. The first step, Reedy fibrancy, is somewhat tedious and so we provide a proper reference.

Lemma 9.36 ([Rez01, Lemma 3.9])**.** *Let $\mathcal{C}$ be a category. Then $\mathcal{N}(\mathcal{C})$ is Reedy fibrant.*

Lemma 9.37. *Let $\mathcal{C}$ be a category. Then the Rezk nerve $\mathcal{N}(\mathcal{C})$ is a Segal space.*

Proof. Let us fix $n \geq 2, l \geq 0$. Following Lemma 9.33, it suffices to prove the map

$$
\begin{array}{c}
\mathrm{Hom}_{\mathcal{C}at}([n] \times I(l), \mathcal{C}) \\
\big\downarrow \\
\mathrm{Hom}_{\mathcal{C}at}([1] \times I(l), \mathcal{C}) \times_{\mathrm{Hom}_{\mathcal{C}at}([0]\times I(l),\mathcal{C})} \cdots \times_{\mathrm{Hom}_{\mathcal{C}at}([0]\times I(l),\mathcal{C})} \mathrm{Hom}_{\mathcal{C}at}([1] \times I(l), \mathcal{C})
\end{array}
$$

is a bijection. This map coincides with the map

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([n], \mathrm{Fun}(I(l), \mathcal{C})) \\ \downarrow \\ \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1], \mathrm{Fun}(I(l), \mathcal{C})) \times_{\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([0], \mathrm{Fun}(I(l), \mathcal{C}))} \cdots \times_{\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([0], \mathrm{Fun}(I(l), \mathcal{C}))} \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1], \mathrm{Fun}(I(l), \mathcal{C})) \end{array}$$

However, this map is a bijection if $N(\mathrm{Fun}(I(l), \mathcal{C}))$ satisfies the Segal condition, which is true by Proposition 2.13. □

Lemma 9.38. *Let $\mathcal{C}$ be a category. An object in the Segal space $\mathcal{N}(\mathcal{C})$ is an object in $\mathcal{C}$. Similarly, a morphism in $\mathcal{N}(\mathcal{C})$ is a morphism in $\mathcal{C}$.*

Proof. Exercise 9.20 □

More generally, we have the following.

Lemma 9.39. *Let $\mathcal{C}$ be a category, and let x, y be two objects in the Segal space $\mathcal{N}(\mathcal{C})$. Then the mapping space $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)$ is isomorphic to the constant space on the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$.*

Proof. By Remark 9.31 and Definition 7.11, the mapping space $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)$ is the pullback of the following diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y) \arrow[r] \arrow[d] & N(\mathrm{Fun}([1], \mathcal{C})^{\simeq}) \arrow[d, two heads, "(0^*, 1^*)"] \\ \mathbb{D}^0 \arrow[r, "(x,y)"] & N\mathcal{C}^{\simeq} \times N\mathcal{C}^{\simeq} \end{tikzcd} .$$

We already saw in Lemma 9.38 that $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(x, y)$. Moreover, we have

$$\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)_l = \{F : [1] \times I(l) \to \mathcal{C} \mid F|_{\{0\} \times I(l)} = \{x\}, F|_{\{1\} \times I(l)} = \{y\}\},$$

where $\{x\}, \{y\}$ are the constant functors $I(l) \to \mathcal{C}$ with value x, resp. y. A functor $F : [1] \times I(l) \to \mathcal{C}$ satisfying this condition necessarily factors through $[1] \times I(0)$, which means it is determined by a morphism in $\mathcal{C}$ from x to y. Thus we have $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)_l \cong \mathrm{Hom}_{\mathcal{C}}(x, y)$ for every l, which means $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)$ is the constant space on the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$. □

Intuition 9.40. An element in $\mathrm{map}_{\mathcal{N}(\mathcal{C})}(x, y)_1$ is a functor $F : [1] \times I(1) \to \mathcal{C}$ of the following form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] & y \arrow[d, "\mathrm{id}_y"] \\ x \arrow[r, "g"'] & y \end{tikzcd} ,$$

which implies $\mathrm{id}_y \circ f = g \circ \mathrm{id}_x$, which means $f = g$.

We can use a similar argument to prove the following analogous result.

Lemma 9.41. *Let $\mathcal{C}$ be a groupoid. Then for two objects x, y in the Kan complex $N\mathcal{C}$, $\mathcal{P}\mathrm{ath}_{N\mathcal{C}}(x, y)$ is isomorphic to the constant space on the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$.*

Proof. Exercise 9.21 □

Proposition 9.42. *Let $\mathcal{C}$ be a category. Then $\mathrm{Ho}(\mathcal{N}(\mathcal{C}))$ is isomorphic to $\mathcal{C}$.*

Proof. By Construction 7.42, the objects of $\mathrm{Ho}(\mathcal{N}(\mathcal{C}))$ are the objects of $\mathcal{N}(\mathcal{C})$, which, following Lemma 9.38, are the objects of $\mathcal{C}$. Now, the morphisms in $\mathrm{Ho}(\mathcal{N}(\mathcal{C}))$ are homotopy classes of morphisms in $\mathcal{N}(\mathcal{C})$. However, by Lemma 9.39, those are precisely the morphisms in $\mathcal{C}$. Thus $\mathrm{Ho}(\mathcal{N}(\mathcal{C}))$ is isomorphic to $\mathcal{C}$. □

Lemma 9.43. *Let $\mathcal{C}$ be a category. Then a morphism in $\mathcal{N}(\mathcal{C})$ is a homotopy equivalence if and only if it is an isomorphism in $\mathcal{C}$.*

Proof. Exercise 9.22 □

Theorem 9.44 (Rezk nerve is a complete Segal space)**.** *Let $\mathcal{C}$ be a category. Then $\mathcal{N}(\mathcal{C})$ is a complete Segal space.*

Proof. We have already shown it is a Segal space (Lemma 9.37), so it suffices to check the completeness condition. Let x, y be two objects in $\mathcal{N}(\mathcal{C})$. By Lemma 9.41, we have a bijection of constant spaces

$$\mathcal{P}\mathrm{ath}_{N\mathcal{C}^{\simeq}}(x,y) \cong \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y).$$

On the other hand, in Lemma 9.39, we showed that we have the equality of constant simplicial sets

$$\mathrm{map}_{\mathcal{N}\mathcal{C}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(x,y).$$

Moreover, by Lemma 9.43, homotopy equivalences in $\mathcal{N}(\mathcal{C})$ are precisely the isomorphisms in $\mathcal{C}$. Hence, we similarly have an equality of constant simplicial sets

$$\mathrm{hoequiv}_{\mathcal{N}\mathcal{C}}(x,y) = \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y),$$

where $\mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y)$ is the constant simplicial set on the set of isomorphisms from x to y in $\mathcal{C}$. Thus, following Proposition 9.22, the completeness condition translates to

$$P_{x,y} \colon \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y) \to \mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y)$$

being a bijection of constant spaces. Note that, under this correspondence, the map $P_{x,y}$ sends an isomorphism from x to y to the same isomorphism regarded as a homotopy equivalence in $\mathcal{N}(\mathcal{C})$. Hence $P_{x,y}$ is the identity map on $\mathrm{Hom}_{\mathcal{C}^{\simeq}}(x,y)$, which is indeed a bijection. □

Exercises.

Exercise 9.1. Let $\mathcal{C}$ be a category and let

$$T = \mathrm{disc}(N\mathcal{C}).$$

Compute $T_{n,l}$ for all $n, l \geq 0$. Then describe explicitly which simplicial direction contains the nerve of $\mathcal{C}$ and which simplicial direction is constant.

Exercise 9.2. Let K be a Kan complex. Explain how the composition of morphisms in

$$cK$$

corresponds to concatenation of paths in K. Why is this composition only well-defined up to homotopy?

Exercise 9.3. Let $\mathcal{C}$ be a category and let

$$T = \mathrm{disc}(N\mathcal{C}).$$

Prove that for every collection of objects

$$x_0, \ldots, x_n \in \mathcal{C},$$

the generalized mapping space

$$\mathrm{map}_T(x_0, \ldots, x_n)$$

is the constant simplicial set associated to the set of strings of composable morphisms

$$x_0 \to x_1 \to \cdots \to x_n$$

in $\mathcal{C}$.

Exercise 9.4. Let $\mathcal{C}$ be a category, let

$$T = \mathrm{disc}(N\mathcal{C}),$$

and let

$$f \colon x \to y, \qquad g \colon y \to z$$

be morphisms in $\mathcal{C}$. Prove that

$$\mathrm{Comp}_T(f,g)$$

is a single point, corresponding to the ordinary composite

$$g \circ f \colon x \to z.$$

Exercise 9.5. Let $\mathcal{C}$ be a category and let
$$T = \mathrm{disc}(N\mathcal{C}).$$
Prove that two morphisms in T are homotopic if and only if the corresponding morphisms in $\mathcal{C}$ are equal.

Exercise 9.6. Prove that for every category $\mathcal{C}$, there is an isomorphism of categories
$$\mathrm{Ho}(\mathrm{disc}(N\mathcal{C})) \cong \mathcal{C}.$$

Exercise 9.7. Let $\mathcal{C}$ be a groupoid. Compare the two Segal spaces
$$\mathrm{disc}(N\mathcal{C}) \qquad \text{and} \qquad c(N\mathcal{C}).$$
What are their objects? What are their mapping spaces? Which one is discrete in the vertical direction?

Exercise 9.8. Recall the notation
$$E(1) = \mathrm{disc}NI(1).$$
Describe explicitly the objects and morphisms of the Segal space $E(1)$. In particular, compute
$$E(1)_0 \qquad \text{and} \qquad E(1)_1$$
as simplicial sets.

Exercise 9.9. Let W be a Segal space. Unpack the map
$$00^* \colon W_0 \to W_1.$$
Explain why its value on an object $x \in W_{00}$ is the identity morphism
$$\mathrm{id}_x \colon x \to x.$$

Exercise 9.10. Let W be a Segal space and let x, y be objects of W. Describe the two spaces
$$\mathcal{P}\mathrm{ath}_{W_0}(x,y) \qquad \text{and} \qquad \mathrm{hoequiv}_W(x,y).$$
Explain the intended meaning of the map
$$P_{x,y} \colon \mathcal{P}\mathrm{ath}_{W_0}(x,y) \to \mathrm{hoequiv}_W(x,y).$$

Exercise 9.11. Unpack the definition of a complete Segal space groupoid. How does it differ from a Segal space groupoid?

Exercise 9.12. Let $\mathcal{C}$ be a category. Unpack the definition of the Rezk nerve
$$\mathcal{N}\mathcal{C}.$$
In particular, write down explicitly the simplicial set
$$(\mathcal{N}\mathcal{C})_n$$
for each $n \geq 0$.

Exercise 9.13. Let $\mathcal{C}$ be a category. Explain the difference between the ordinary nerve Segal space
$$\mathrm{disc}N\mathcal{C}$$
and the Rezk nerve
$$\mathcal{N}\mathcal{C}.$$
In particular, compare their object spaces.

Exercise 9.14. Use Proposition 7.33 to prove Lemma 9.12: For every x in W_0, id_x is a homotopy equivalence.

Exercise 9.15. Prove Lemma 9.17: For a given Segal space W, use Lemma 8.25 and Reedy fibrancy to show that
$$W_{\mathrm{hoequiv}} \to W_0 \times W_0$$
is a Kan fibration.

Exercise 9.16. Let W be a complete Segal space groupoid. Prove that

$$W_{\text{hoequiv}} = W_1.$$

Then use completeness to show that

$$00^*: W_0 \to W_1$$

is an equivalence.

Exercise 9.17. Let W be a complete Segal space groupoid. Prove that the maps

$$0^*, 1^*: W_1 \to W_0$$

are equivalences. Then explain why they are trivial Kan fibrations.

Exercise 9.18. Let W be a complete Segal space groupoid. Prove that the projection

$$W_1 \times_{W_0} W_1 \to W_1$$

is a trivial Kan fibration.

Exercise 9.19. Prove Proposition 9.27: Use the fact that $cK \simeq \text{const}K$ and the previous three exercises to prove that a complete Segal space W is a complete Segal space groupoid if and only if W is equivalent to cW_0.

Exercise 9.20. Using Lemma 9.33, prove Lemma 9.38: objects and morphisms in the Segal space

$$\mathcal{N}(\mathcal{C})$$

are precisely objects and morphisms of the category $\mathcal{C}$.

Exercise 9.21. Prove Lemma 9.41: if $\mathcal{C}$ is a groupoid, then for two objects x, y in $N\mathcal{C}$,

$$\mathcal{P}\text{ath}_{N\mathcal{C}}(x, y)$$

is the constant simplicial set associated to

$$\text{Hom}_{\mathcal{C}}(x, y).$$

Exercise 9.22. Use Proposition 8.6 to prove Lemma 9.43: a morphism in

$$\mathcal{N}(\mathcal{C})$$

is a homotopy equivalence if and only if the corresponding morphism in $\mathcal{C}$ is an isomorphism.

Exercise 9.23. Compute the Segal space

$$E(1) = \text{disc}NI(1)$$

in horizontal degrees 0, 1, and 2. Identify all non-degenerate horizontal simplices.

Exercise 9.24. Compute

$$E(1)_{\text{hoequiv}}.$$

Then compute the map

$$00^*: E(1)_0 \to E(1)_{\text{hoequiv}}$$

and show directly that it is not an equivalence.

Exercise 9.25. Compute the map

$$P_{0,1}: \mathcal{P}\text{ath}_{E(1)_0}(0, 1) \to \text{hoequiv}_{E(1)}(0, 1).$$

Explain why this map is not an equivalence.

Exercise 9.26. Prove that $\mathcal{N}[0]$ is isomorphic to Δ^0.

Exercise 9.27. Prove that $\mathcal{N}[1]$ is isomorphic to Δ^1.

Exercise 9.28. Let $\mathcal{C} = I(1)$. Compute

$$(\mathcal{N}I(1))_0.$$

Compare this object space with the object space of

$$E(1) = \text{disc}NI(1).$$

Exercise 9.29. Let $\mathcal{C}$ be a discrete category associated to a set S. Compute

$$\mathcal{N}\mathcal{C}.$$

What are its object space, morphism space, mapping spaces, and homotopy category?

Exercise 9.30. Let G be a group, regarded as a one-object category. Compute

$$\mathcal{N}G_0 \qquad \text{and} \qquad \mathcal{N}G_1.$$

Then compute

$$\mathrm{Ho}(\mathcal{N}G).$$

Exercise 9.31. Let $\mathcal{C}$ be a category and let x, y be objects of $\mathcal{C}$. Compute

$$\mathcal{P}\mathrm{ath}_{\mathcal{N}\mathcal{C}^{\simeq}}(x, y).$$

Show that it is empty if x and y are not isomorphic, and is the constant simplicial set on the set of isomorphisms $x \to y$ if they are isomorphic.

Exercise 9.32. Let $\mathcal{C}$ be a category and let x, y be objects. Compute

$$\mathrm{map}_{\mathcal{N}\mathcal{C}}(x, y)_0 \qquad \text{and} \qquad \mathrm{map}_{\mathcal{N}\mathcal{C}}(x, y)_1.$$

Use the computation in degree 1 to explain why homotopic morphisms in $\mathcal{N}\mathcal{C}$ are equal.

Exercise 9.33. Let $\mathcal{C}$ be a category and let

$$f \colon x \to y$$

be a morphism. Compute when f is a homotopy equivalence in

$$\mathcal{N}\mathcal{C}.$$

Then compare this with the corresponding statement for

$$\mathrm{disc}N\mathcal{C}.$$

Exercise 9.34. Let $\mathcal{C}$ be a category with two isomorphic objects x, y. Compare the object spaces of

$$\mathrm{disc}N\mathcal{C} \qquad \text{and} \qquad \mathcal{N}\mathcal{C}.$$

Explain how the Rezk nerve adds a path between x and y in the object space.

Exercise 9.35. Let $\mathcal{C}$ be a groupoid. Compare the Rezk nerve

$$\mathcal{N}\mathcal{C}$$

with the Segal space

$$c(N\mathcal{C}).$$

Compute both in bidegrees $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$.

Exercise 9.36. Let $\mathcal{C}$ be a category and let

$$f \colon x \to y, \qquad g \colon y \to z$$

be composable morphisms. Compute

$$\mathrm{Comp}_{\mathcal{N}\mathcal{C}}(f, g).$$

Show that it is a point corresponding to the ordinary composite

$$g \circ f.$$

10. Interlude: Another Look at the Yoneda Lemma

Until now we have understood *what* a higher category is: a complete Segal space. As a next question we ask what we can *do* with it. One major result in classical category theory is the *Yoneda lemma* which characterizes natural transformations out of representable functors. Unsurprisingly the Yoneda lemma has an analogue in the higher categorical setting. Unfortunately this generalization is not as straightforward as one might initially anticipate. Hence, in this section we first review the many faces of the Yoneda lemma in the classical setting.

While the Yoneda lemma is very famous, it is a lesser known aspect of the Yoneda lemma is that it can be expressed in several different ways. Concretely we want to review four different faces of the Yoneda lemma, which are summarized in this table:

	Hom	*Tensor*
Functor	$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c,-),F) \xrightarrow{\cong} F(c)$ Lemma 10.1	$\mathrm{Hom}_{\mathcal{C}}(c,-) \underset{\mathcal{C}}{\otimes} F \xrightarrow{\cong} F(c)$ Lemma 10.5
Fibration	$\mathrm{Hom}_{/\mathcal{C}}(\mathcal{C}_{/c},\mathcal{D}) \xrightarrow{\cong} \{c\} \underset{\mathcal{C}}{\times} \mathcal{D}$ Lemma 10.27	$\mathcal{C}_{c/} \underset{\mathcal{C}}{\otimes} \mathcal{D} \xrightarrow{\cong} \{c\} \underset{\mathcal{C}}{\otimes} \mathcal{D}$ Lemma 10.28

Let us start with the most common form of the Yoneda lemma, which can be found in any introductory book on classical category theory. Here is a version that appears in [ML98, Page 61].

Lemma 10.1. *(Hom version of Yoneda for functors) If $F\colon \mathcal{C} \to Set$ is a functor and $c \in \mathcal{C}$ an object, then the natural map*

$$\mathfrak{Yon}\colon\ \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c,-),F) \xrightarrow{\quad\cong\quad} F(C)$$

$$[\alpha\colon \mathrm{Hom}_{\mathcal{C}}(c,-) \to F] \longmapsto \alpha_c(\mathrm{id}_c)$$

is a bijection.

Proof. Let us first construct the inverse. Given $x \in F(C)$, let $\mathfrak{Com}(x)\colon \mathrm{Hom}_{\mathcal{C}}(c,-) \to F$ be the natural transformation given by

$$\mathfrak{Com}(x)_d(f\colon c \to d) := F(f)(x),$$

which is natural by Exercise 10.1.

Now, as F is a functor,

$$\mathfrak{Yon}\mathfrak{Com}(x) = \mathfrak{Com}(x)_c(\mathrm{id}_c) = F(\mathrm{id}_c)(x) = x.$$

On the other side, for a given $f\colon c \to d$,

$$\mathfrak{Com}\mathfrak{Yon}(\alpha)_d(f) = F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f),$$

where the last equality follows from naturality, given in Exercise 10.2. □

Intuition 10.2. The last equality of the proof (and also the content of Exercise 10.2) is the crux of the proof. By effectively using the composition in $\mathcal{C}$, the value of a natural transformation $\alpha_d(f)$ is given by $F(f)(\alpha_c(\mathrm{id}_c))$ and hence fully determined by $\alpha_c(\mathrm{id}_c)$.

There is, however, a different way this equivalence can be phrased. It relies on the *Hom-Tensor Adjunction*.

10.1. **Tensor Product of Functors and Yoneda Lemma.**

Most of the material in this subsection can be found in greater detail in [MLM94, VII.2]. For this subsection let $\mathcal{C}$ be a fixed category and $F\colon \mathcal{C} \to Set$ and $P\colon \mathcal{C}^{op} \to \mathcal{S}et$ be two functors. Then we define the tensor product as the following colimit diagram $F \otimes_{\mathcal{C}} P$

$$\coprod_{c,c' \in \mathcal{C}} P(c) \times \mathrm{Hom}_{\mathcal{C}}(c,c') \times F(c) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} P(c) \times F(c) \longrightarrow F \underset{\mathcal{C}}{\otimes} P\ ,$$

where $\varphi(a, f, b) = (P(f)(a), b)$ and $\psi(a, f, b) = (a, F(f)(b))$. So the tensor product of two functors is the product of the values quotiented out by the mapping relations. This definition generalizes the tensor product of a right and left module over a ring, which is the motivation for this notation. Similar to the case of rings this definition of a tensor product fits into a *hom-tensor adjunction*.

Theorem 10.3. *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{S}et$ a functor. Then we have the adjunction*

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}et) \underset{\mathrm{Hom}_{\mathcal{S}et}(F(-),-)}{\overset{F\otimes_{\mathcal{C}}-}{\rightleftarrows}} \mathcal{S}et$$

where the left adjoint takes P to $P \otimes_{\mathcal{C}} F$ and the right adjoint takes a set S to the functor which takes an object C to $Hom_{\mathcal{S}et}(F(c), S)$.

Remark 10.4. Note that we could have made the same construction for the case where $\mathcal{S}et$ is replaced with any category which has all colimits. However, here we do not need to work at this level of generality. For more details on the general construction see [MLM94, Page 358].

With the tensor product at hand we can state another version of the Yoneda lemma.

Lemma 10.5. *(Tensor version of Yoneda for functors) If $P \colon \mathcal{C}^{op} \to Set$ is a functor and $C \in \mathcal{C}$ an object, then the natural map*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} P & \xrightarrow{\cong} & P(c) \\ (f \colon c \to c', a \in P(c')) & \longmapsto & P(f)(a) \end{array}$$

is a bijection.

Proof. Exercise 10.3 □

This version of the Yoneda lemma has the following basic corollaries, which should look quite familiar.

Corollary 10.6. *Let $\mathcal{C}$ be a category and C, C' two objects. Then we have the following isomorphism.*

$$\mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} \mathrm{Hom}_{\mathcal{C}}(-, c') \cong \mathrm{Hom}_{\mathcal{C}}(c, c')$$

Proof. Exercise 10.4 □

Corollary 10.7. *Let $\mathcal{C}$ be a category and $P, Q \colon \mathcal{C}^{op} \to Set$ be two functors. Then a natural transformation $\alpha \colon P \to Q$ is a natural isomorphism if and only if*

$$\mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} P \to \mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} Q$$

is a bijection for every object $C \in \mathcal{C}$.

Proof. Exercise 10.5 □

10.2. **From Functors to Fibrations: The Grothendieck Construction.** We want to now translate the Yoneda lemma from a statement about set-valued functors to a statement about fibrations. This requires us to translate between functors and fibrations which we will do via the Grothendieck construction.

Remark 10.8. We will need a careful understanding of the Grothendieck construction in the coming sections. Thus the review in this section is self-contained. However, the ideas are in no way new and a more detailed approach can be found in many places, such as [MLM94, I.5], or [Joh02, A1.1.7, B1.3.1].

Definition 10.9. Let $\mathcal{C}$ be a category. Define

$$\int_{\mathcal{C}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}et) \to \mathcal{C}at_{/\mathcal{C}}$$

as the functor that takes $F \colon \mathcal{C} \to \mathcal{S}et$ to the category $\int_{\mathcal{C}} F \to \mathcal{C}$ with

- Objects: Pairs (c, x) where c is an object in $\mathcal{C}$ and $x \in F(c)$.
- Morphisms: For two objects (c, x), (d, y) we have

$$\mathrm{Hom}_{\int_{\mathcal{C}} F}((c,x),(d,y)) = \{f \in \mathrm{Hom}_{\mathcal{C}}(c,d) \colon F(f)(x) = y\}.$$

It comes with an evident projection map $\pi_F \colon \int_{\mathcal{C}} F \to \mathcal{C}$. Moreover, for a natural transformation $\alpha \colon F \Rightarrow G$, the functor $\int_{\mathcal{C}} \alpha \colon \int_{\mathcal{C}} F \to \int_{\mathcal{C}} G$ is given as $(c, x) \mapsto (c, \alpha_c x)$.

Notice, by construction the fiber of $\pi_F \colon \int_{\mathcal{C}} F \to \mathcal{C}$ over an object c is the discrete category with object set $F(c)$. This functor has a left adjoint and a right adjoint that we want to define in detail.

Definition 10.10. Let $\mathcal{C}$ be a category. We define the functor

$$\mathcal{C}_{/-} \colon \mathcal{C} \to \mathcal{C}\mathrm{at}_{/\mathcal{C}}$$

that takes an object to the over-category $\mathcal{C}_{/c}$ and a morphism $f \colon c \to d$ to the post-composition $f_! \colon \mathcal{C}_{/c} \to \mathcal{C}_{/d}$.

Similarly, define the functor

$$\mathcal{C}_{-/} \colon \mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_{/\mathcal{C}}$$

that takes an object to the under-category $\mathcal{C}_{c/}$ and a morphism $f \colon c \to d$ to the precomposition $f^* \colon \mathcal{C}_{d/} \to \mathcal{C}_{c/}$.

For a given category over $\mathcal{C}$, $p \colon \mathcal{D} \to \mathcal{C}$ define

$$\mathcal{T}_{\mathcal{C}}(p \colon \mathcal{D} \to \mathcal{C}) \colon \mathcal{C} \to \mathcal{S}\mathrm{et}$$

as the composition

$$\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \mathcal{C}\mathrm{at}_{/\mathcal{C}} \xrightarrow{- \times_{\mathcal{C}} \mathcal{D}} \mathcal{C}\mathrm{at} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$$

in other words we have $\mathcal{T}_{\mathcal{C}}(c) = \pi_0(\mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{D})$. Similarly, define

$$\mathcal{H}_{\mathcal{C}}(p \colon \mathcal{D} \to \mathcal{C}) \colon \mathcal{C} \to \mathcal{S}\mathrm{et}$$

as the composition

$$\mathcal{C} \xrightarrow{(\mathcal{C}_{-/})^{op}} (\mathcal{C}\mathrm{at}_{/\mathcal{C}})^{op} \xrightarrow{\mathrm{Hom}_{/\mathcal{C}}(-,\mathcal{D})} \mathcal{S}\mathrm{et}$$

meaning we have $\mathcal{H}_{\mathcal{C}}(c) = \mathrm{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D})$. We claim that $\mathcal{T}_{\mathcal{C}}$ is the left adjoint and $\mathcal{H}_{\mathcal{C}}$ is the right adjoint to $\int_{\mathcal{C}}$.

Proposition 10.11. *We have the following diagram of adjunctions*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \arrow[r, "\int_{\mathcal{C}}" description] & \mathcal{C}\mathrm{at}_{/\mathcal{C}} \arrow[l, bend right, "\mathcal{T}_{\mathcal{C}}"', "\perp"] \arrow[l, bend left, "\mathcal{H}_{\mathcal{C}}", "\perp"'] \end{tikzcd}.$$

Proof. We first prove the right adjoint. First of all notice $\int_{\mathcal{C}}$ commutes with colimits. Indeed, for a given diagram $G \colon I \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ it is direct computation that the induced cocone $\int_{\mathcal{C}} G_i \to \int_{\mathcal{C}} \mathrm{colim}_I G_i$ satisfies the universal property of the universal cocone. Now, by [MLM94, Corollary I.5.4], every colimit preserving functor out of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is the left Kan extension of its restriction to $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, meaning we have the following left Kan extension

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "\mathcal{C}_{-/}"] \arrow[d, "\mathcal{Y}\mathrm{on}"'] & \mathcal{C}\mathrm{at}_{/\mathcal{C}} \\ \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \arrow[ur, dashed, "\int_{\mathcal{C}}"'] & \end{tikzcd}$$

which means it has a right adjoint given by $\mathcal{H}_{\mathcal{C}}(p \colon \mathcal{D} \to \mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_{-/}, \mathcal{D})$.

For the left adjoint, we first show that $\mathcal{T}_\mathcal{C}$ commutes with colimits. As colimits are computed point-wise this means we have to prove that for every object c the functor

$$\pi_0(\mathcal{C}_{/c} \times_\mathcal{C} -) \colon \mathcal{C}\mathrm{at}_{/\mathcal{C}} \to \mathcal{S}\mathrm{et}$$

commutes with colimits. This functor is a composition and so we check separately that both are left adjoints:

(1) $\mathcal{C}_{/c} \times_\mathcal{C} -$ is a left adjoint because $\mathcal{C}_{/c} \to \mathcal{C}$ is a Conduché functor [Con72].
(2) π_0 is the left adjoint of the inclusion functor $\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$.

Now, we prove that $\mathcal{T}_\mathcal{C}$ is the left adjoint. Recall that $[n]$ is the category given via the poset structure $\{0 \leq 1 \leq ... \leq n\}$ and that every category over $\mathcal{C}$ is a colimit of functors $\alpha \colon [n] \to \mathcal{C}$ and so the result follows from the following natural isomorphisms:

$$\begin{array}{l} \mathrm{Hom}_{/\mathcal{C}}(\alpha\colon [n] \to \mathcal{C}, \textstyle\int_\mathcal{C} G) \xrightarrow[\cong]{\{0\}^*} \mathrm{Hom}_{/\mathcal{C}}(\alpha \circ \{0\}\colon [0] \to \mathcal{C}, \textstyle\int_\mathcal{C} G) \xrightarrow[\cong]{\alpha(0)} G(\alpha(0)) \xrightarrow[\cong]{\mathcal{Y}\mathrm{on}_{\alpha(0)}} \\ \mathrm{Nat}(\mathrm{Hom}(\alpha(0), -), G) \xrightarrow{\cong} \mathrm{Nat}(\mathcal{T}_\mathcal{C}(\alpha\colon [0] \to \mathcal{C}), G) \xrightarrow{\cong} \mathrm{Nat}(\mathcal{T}_\mathcal{C}(\alpha\colon [n] \to \mathcal{C}), G) \end{array}.$$

Let us explain the various natural isomorphisms that require a justification.

Notice a morphism in $\int_\mathcal{C} G$ is of the form $f \colon (c, x) \to (d, G(f)(x))$, where $f \colon c \to d$ is a morphism in $\mathcal{C}$. Hence, for a given functor $\alpha \colon [n] \to \mathcal{C}$, which we can depict by a chain of n morphisms in $\mathcal{C}$, $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} ... \xrightarrow{f_n} c_n$, a functor $[n] \to \int_\mathcal{C} G$ over $\mathcal{C}$ that lifts α is of the form $(c_0, x) \xrightarrow{f_1} (c_1, G(f_1)(x)) \xrightarrow{f_2} ... \xrightarrow{f_n} (c_1, G(f_n \circ ... \circ f_2 \circ f_1)(x))$, meaning it is uniquely determined by the value of the object 0 in $[n]$.

The third isomorphism is the Yoneda lemma. The fourth isomorphism follows from the fact that $\mathcal{T}_\mathcal{C}(\alpha \colon [0] \to \mathcal{C}) = \pi_0(\mathcal{C}_{/c} \times_\mathcal{C} [0]) \cong \pi_0(\mathrm{Hom}(\alpha(0), c)) = \mathrm{Hom}(\alpha(0), c)$. Finally, for the last isomorphism we observe that two objects of the form $\alpha(0) \to \alpha(1) \to ... \to \alpha(n) \to c$ in $\mathcal{C}_{/c} \times_\mathcal{C} [n]$ are in the same path-component if they compose to the same morphism $\alpha(0) \to c$ giving us the desired bijection $\mathcal{T}_\mathcal{C}(\alpha \circ \{0\}^* \colon [0] \to \mathcal{C}) \cong \mathcal{T}_\mathcal{C}(\alpha \colon [n] \to \mathcal{C}) = \pi_0(\mathcal{C}_{/c} \times_\mathcal{C} [n])$. □

In fact $\int_\mathcal{C}$ has even more desirable properties.

Lemma 10.12. *$\int_\mathcal{C} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{at}_{/\mathcal{C}}$ is fully faithful.*

Proof. Let $F, G \colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ be two functors. We need to prove that the map

$$\mathrm{Nat}(F, G) \to \mathrm{Hom}_{/\mathcal{C}}(\int_\mathcal{C} F, \int_\mathcal{C} G)$$

is a bijection of sets. For that we will construct an inverse. Concretely, we define

$$\mathcal{I}_\mathcal{C} \colon \mathrm{Hom}_{/\mathcal{C}}(\int_\mathcal{C} F, \int_\mathcal{C} G) \to \mathrm{Nat}(F, G)$$

as follows. For a given functor $H \colon \int_\mathcal{C} F \to \int_\mathcal{C} G$ over $\mathcal{C}$ we define the natural transformation $\mathcal{I}_\mathcal{C}(H)_c(x) = H(c, x)$. The functoriality of H implies that $\mathcal{I}(H)$ is natural.

It remains to show these are inverses. For a given natural transformation$\alpha \colon F \Rightarrow G$ we have

$$(\mathcal{I}_\mathcal{C} \int_\mathcal{C} (\alpha))_c(x) = \int_\mathcal{C} (\alpha)(c, x) = \alpha_c(x)$$

and on the other side

$$\int_\mathcal{C} (\mathcal{I}_\mathcal{C}(H))(c, x) = (\mathcal{I}_\mathcal{C}(H))_c(x) = H(c, x)$$

finishing the proof. □

This has a direct implication for $\mathcal{T}_\mathcal{C}$ and $\mathcal{H}_\mathcal{C}$.

Corollary 10.13. *$\mathcal{T}_\mathcal{C}$ is a localization functor and $\mathcal{H}_\mathcal{C}$ is a colocalization functor.*

We end this subsection by observing that while $\int_\mathcal{C}$ is fully faithful, it is in fact not essentially surjective.

Definition 10.14. A functor $p\colon \mathcal{D} \to \mathcal{C}$ is *conservative* if it reflects isomorphisms.

Lemma 10.15. *Let $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ be a functor. Then $\pi_F\colon \int_{\mathcal{C}} F \to \mathcal{C}$ is conservative.*

Proof. Exercise 10.6 □

Thus we need to restrict our attention to the essential image of $\int_{\mathcal{C}}$, which leads us to discrete Grothendieck fibrations.

10.3. Yoneda Lemma for Grothendieck Fibrations.

In this subsection we want to use the fully faithful functor $\int_{\mathcal{C}}$ to translate both versions of the Yoneda lemma from a functorial statement to a fibrational one.

This might not be a major improvement when studying 1-categories, however, in the world of higher categories functors can be difficult to study, because of the homotopy coherence. On the other hand, fibrations can be defined and studied in a straightforward manner. Thus a fibrational approach to the Yoneda lemma is an excellent first step for a generalization to a Yoneda lemma for simplicial spaces.

As we observed in Lemma 10.12, $\int_{\mathcal{C}}$ is fully faithful, however, it is not essentially surjective!

Definition 10.16. A functor $P\colon \mathcal{D} \to \mathcal{C}$ is a *discrete Grothendieck opfibration* over $\mathcal{C}$ if it is in the essential image of $\int_{\mathcal{C}}$, meaning there exists a functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ and isomorphism $\int_{\mathcal{C}} F \xrightarrow{\cong} \mathcal{D}$ over $\mathcal{C}$.

Fortunately, there is also an internal characterization of discrete Grothendieck opfibrations.

Lemma 10.17. *A functor $P\colon \mathcal{D} \to \mathcal{C}$ is a discrete Grothendieck opfibration over $\mathcal{C}$ if and only if for any map $f\colon c \to c'$ in $\mathcal{C}$ and object d in $\mathcal{D}$ such that $P(d) = c$, there exists a* unique *lift $\hat{f}\colon d \to d'$ such that $P(\hat{f}) = f$.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ be a functor. Then $\int_{\mathcal{C}} F \to \mathcal{C}$ satisfies the lifting condition stated in the lemma. Indeed, for a morphism $f\colon c \to c'$ and a lift (c, x) where $x \in F(c)$, there is a unique lift given by the morphism $f\colon (c, x) \to (c', F(f)(x))$.

On the other hand let us assume that $P\colon \mathcal{D} \to \mathcal{C}$ satisfies the lifting condition of the lemma. Then we will construct a functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $\int_{\mathcal{C}} F \cong \mathcal{D}$ over $\mathcal{C}$.

First note that the unique lifting condition implies that the fiber of P over every given point c, $P^{-1}(c)$, is a discrete category i.e. a set. Indeed let f be a morphism in $\mathcal{D}$ such that $P(f) = \mathrm{id}_c$. Then by the uniqueness assumption $f = \mathrm{id}$.

Now define F as follows:

- **Objects:** For an object c in $\mathcal{C}$ define $F(c) = P^{-1}(c)$
- **Morphisms:** For a morphism $f\colon c \to c'$ define $F(f)\colon F(c) \to F(c')$ as the map that takes $x \in F(c)$ to the target of the unique lift of f in $\mathcal{D}$.

The standard projection $\pi\colon \int_{\mathcal{C}} F \to \mathcal{C}$ exactly recovers $P\colon \mathcal{D} \to \mathcal{C}$. □

Discrete Grothendieck opfibrations admit several alternative characterizations.

Lemma 10.18. *Let $P\colon \mathcal{D} \to \mathcal{C}$ be a functor. Then the following are equivalent:*

(1) P is a discrete Grothendieck opfibration.
(2) The following square is a pullback square of sets

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}} \arrow[r, "source"] \arrow[d, "F"] & \mathrm{Obj}_{\mathcal{D}} \arrow[d, "F"] \\ \mathrm{Mor}_{\mathcal{C}} \arrow[r, "source"] & \mathrm{Obj}_{\mathcal{C}} \end{tikzcd} \tag{10.19}$$

(3) *The following square is a pullback square of categories*

$$\begin{tikzcd} \mathrm{Fun}([1],\mathcal{D}) \arrow[r, "0^*"] \arrow[d, "\mathrm{Fun}([1],F)"] & \mathcal{D} \arrow[d, "F"] \\ \mathrm{Fun}([1],\mathcal{C}) \arrow[r, "0^*"] & \mathcal{C} \end{tikzcd} \tag{10.20}$$

Proof. Exercise 10.7 □

The previous characterization allows us to give a contravariant version of Grothendieck opfibrations.

Definition 10.21. $P\colon \mathcal{D} \to \mathcal{C}$ is called a *discrete Grothendieck fibration* if for any map $f\colon c \to c'$ in $\mathcal{C}$ and object d' in $\mathcal{D}$ such that $P(d') = c'$, there exists a *unique* lift $\hat{f}\colon d \to d'$ such that $P(\hat{f}) = f$.

Lemma 10.22. *Let $p\colon \mathcal{D} \to \mathcal{C}$ be a discrete Grothendieck fibration, corresponding to the functor $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$. Then p is also a discrete Grothendieck opfibration if and only if $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ takes every morphism to an isomorphism.*

Proof. Exercise 10.8 □

We can prove further facts of interest about discrete Grothendieck fibrations.

Lemma 10.23. *In the following pullback square of categories*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "Q"] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{D} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

if P is a discrete Grothendieck fibration, then Q is also a discrete Grothendieck fibration.

Proof. Exercise 10.9 □

Lemma 10.24. *Let $P\colon \mathcal{D} \to \mathcal{C}$ and $Q\colon \mathcal{E} \to \mathcal{D}$ be discrete Grothendieck fibrations. Then the composition $P \circ Q\colon \mathcal{E} \to \mathcal{C}$ is also a discrete Grothendieck fibration.*

Proof. Exercise 10.10 □

Remark 10.25. Discrete Grothendieck fibrations are a special case of more general *Grothendieck fibrations* that correspond to functors valued in categories. See [Str18] for a readable introduction to general Grothendieck fibrations.

We now want to move on to the Yoneda lemma for discrete Grothendieck fibrations, but for that we need the analogue of representable functors.

Example 10.26. Let us determine the category $\int_{\mathcal{C}} \mathrm{Hom}(c, -)$. Its objects are pairs $(d, f\colon c \to d)$ and a morphism $(d, f\colon c \to d) \to (d', f'\colon c \to d')$ is a morphism $g\colon d \to d'$ such that $gf = f'\colon c \to d'$. Thus we just rediscovered the projection from the under-category $\mathcal{C}_{c/} \to \mathcal{C}$.

With the previous remarks at hand we can now phrase the first fibered version of the Yoneda Lemma.

Lemma 10.27. *(Hom version of Yoneda for fibered categories) Let $P\colon \mathcal{D} \to \mathcal{C}$ be a discrete Grothendieck opfibration. Then the natural map between the sets of functors*

$$\begin{array}{ccc} \mathrm{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D}) & \xrightarrow{\cong} & \mathrm{Hom}_{/\mathcal{C}}(\{c\}, \mathcal{D}) = \{c\} \underset{\mathcal{C}}{\times} \mathcal{D} \\ [F\colon \mathcal{C}_{c/} \to \mathcal{D}] & \longmapsto & F(\mathrm{id}_c) \end{array}$$

is a bijection. Here $\mathrm{Hom}_{/\mathcal{C}}$ denotes the hom set in the category $\mathcal{C}\mathrm{at}_{/\mathcal{C}}$ and $\mathrm{id}_c\colon c \to c$ is seen as an object in $\mathcal{C}_{c/}$.

Proof. Exercise 10.11 □

Similar to the previous part we also have a tensor version of the Yoneda lemma for fibered categories. First, however, we have to define a notion of tensor product for fibered categories. Recall that the tensor product of two functors $F\colon \mathcal{C} \to \mathcal{S}et$, $P\colon \mathcal{C}^{op} \to \mathcal{S}et$ is given by a quotient on the set $\coprod_{c \in \mathcal{C}} F(c) \times P(c)$, which reminds us of a fibered product. Hence the fibrational analogue of the tensor product of a discrete Grothendieck opfibration $\mathcal{D} \to \mathcal{C}$ and discrete Grothendieck fibration $\mathcal{E} \to \mathcal{C}$ is given by

$$\mathcal{D} \underset{\mathcal{C}}{\otimes} \mathcal{E} = \pi_0(\mathcal{D} \underset{\mathcal{C}}{\times} \mathcal{E})$$

where π_0 is the set of connected components. With this definition we can state our last version of the Yoneda lemma.

Lemma 10.28. *(Tensor version of Yoneda for fibered categories) Let $P\colon \mathcal{D} \to \mathcal{C}$ be a discrete Grothendieck opfibration. Then the natural map*

$$\begin{array}{ccc} \mathcal{C}_{/c} \underset{\mathcal{C}}{\otimes} \mathcal{D} & \xrightarrow{\cong} & \{c\} \underset{\mathcal{C}}{\times} \mathcal{D} \\ [(f\colon c' \to c, d')] & \longmapsto & Codomain(\hat{f}) \end{array}$$

is a bijection (here $\hat{f}$ is the unique lift of f with domain d').

Proof. Exercise 10.12 □

Our goal in the coming sections is to build the necessary machinery to generalize these statements to the setting of simplicial spaces. In particular, we will define the correct analogue to discrete Grothendieck opfibrations, study their properties and prove the Yoneda lemma.

Exercises.

Exercise 10.1. Complete the first step of Lemma 10.1: Prove that $\mathcal{C}om(x)$ is indeed a natural transformation.

Exercise 10.2. Complete the second step of Lemma 10.1: Prove that for every natural transformation $\alpha\colon \mathrm{Hom}_{\mathcal{C}}(c, -) \to F$ and every morphism $f\colon c \to d$, we have

$$\alpha_d(f) = F(f)(\alpha_c(\mathrm{id}_c)).$$

Exercise 10.3. Prove Lemma 10.5: Use the universal characterization of the tensor product in Theorem 10.3 and the classical Yoneda lemma to show that the map

$$\mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} P \to P(c)$$

is a bijection.

Exercise 10.4. Prove Corollary 10.6: Use Lemma 10.5 to show that the map

$$\mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} \mathrm{Hom}_{\mathcal{C}}(-, c') \to \mathrm{Hom}_{\mathcal{C}}(c, c')$$

is a bijection.

Exercise 10.5. Prove Corollary 10.7: Use Lemma 10.5 to show that a natural transformation $\alpha\colon P \to Q$ is a natural isomorphism if and only if

$$\mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} P \to \mathrm{Hom}_{\mathcal{C}}(c, -) \underset{\mathcal{C}}{\otimes} Q$$

is a bijection for every object $C \in \mathcal{C}$.

Exercise 10.6. Prove Lemma 10.15: Prove that the projection functor $\pi_F\colon \int_{\mathcal{C}} F \to \mathcal{C}$ is conservative by showing that a morphism $f\colon (c, x) \to (d, y)$ is an isomorphism if the underlying morphism $\pi_F(f)\colon c \to d$ is an isomorphism.

Exercise 10.7. Prove Lemma 10.18: Prove a functor is a discrete Grothendieck opfibration if and only if either of the squares Equations (10.19) and (10.20) are pullback squares.

Exercise 10.8. Prove Lemma 10.22: For a discrete Grothendieck fibration $p\colon \mathcal{D} \to \mathcal{C}$ with corresponding functor $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, p is a discrete Grothendieck opfibration if and only if $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ takes every morphism to an isomorphism.

Exercise 10.9. Prove Lemma 10.23: Discrete Grothendieck fibrations are stable under pullback.

Exercise 10.10. Prove Lemma 10.24: If $P\colon \mathcal{D} \to \mathcal{C}$ and $Q\colon \mathcal{E} \to \mathcal{D}$ are discrete Grothendieck fibrations, then the composition $P \circ Q\colon \mathcal{E} \to \mathcal{C}$ is also a discrete Grothendieck fibration.

Exercise 10.11. Prove Lemma 10.27: Use the fully faithfulness of the Grothendieck construction (Lemma 10.12), to show that the map

$$\mathrm{Hom}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D}) \to \mathrm{Hom}_{/\mathcal{C}}(\{c\}, \mathcal{D})$$

is a bijection.

Exercise 10.12. Prove Lemma 10.28: Prove that the map

$$\mathcal{C}_{/c} \underset{\mathcal{C}}{\otimes} \mathcal{D} \to \{c\} \underset{\mathcal{C}}{\times} \mathcal{D}$$

is a bijection.

Exercise 10.13. Let $F\colon \mathcal{D} \to [0]$ be a functor. Show that the following are equivalent:

(1) F is a discrete Grothendieck fibration.
(2) F is a discrete Grothendieck opfibration.
(3) Every morphism in $\mathcal{D}$ is the identity.

Exercise 10.14. Let $P\colon \mathcal{D} \to \mathcal{C}$ be a discrete Grothendieck opfibration. Show that in the following pullback square of categories

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "P"] \\ {[0]} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

the category $\mathcal{F}$ has no non-trivial morphisms.

Exercise 10.15. Let $P\colon \mathcal{D} \to [1]$ be a discrete Grothendieck opfibration. Directly use the lifting condition to construct a set function from the fiber of P over 0 to the fiber of P over 1.

Exercise 10.16. Let $P\colon \mathcal{D} \to [1]$ be a discrete Grothendieck fibration. Directly use the lifting condition to construct a set function from the fiber of P over 1 to the fiber of P over 0.

Exercise 10.17. Let $P\colon \mathcal{D} \to \mathcal{C}$ be a discrete Grothendieck fibration and assume that $\mathcal{D}$ has a terminal object t, such that $P(t) = c$. Prove there is an isomorphism of categories $\mathcal{D} \cong \mathcal{C}_{/c}$.

Exercise 10.18. Let $0\colon [0] \to [1]$. Prove or give a counterexample:

(1) 0 is a discrete Grothendieck fibration.
(2) 0 is a discrete Grothendieck opfibration.

Exercise 10.19. Let $1\colon [0] \to [1]$. Prove or give a counterexample:

(1) 1 is a discrete Grothendieck fibration.
(2) 1 is a discrete Grothendieck opfibration.

11. Left Fibrations and the Yoneda Lemma

Having reviewed the classical story, we now move on to the higher categories. Given the technical nature of this subject, we focus on general result. For further details and proofs for fibrations in the context of complete Segal spaces, we refer the reader to [Ras23].

Let W be a Segal space and x be an object therein. For every object y, we can define a mapping space $\mathrm{map}_W(x, y)$ (Definition 7.11). Ideally, these should assemble into a functor. However, here we encounter a real challenge: Composition is only defined up to homotopy.

Concretely, given $f: y \to z$, $g: z \to w$, we saw in Lemma 7.35 that there are maps

$$g_* \circ f_*, (g \circ f)_* : \mathrm{map}_W(x,y) \to \mathrm{map}_W(x,w)$$

that are equivalent, but not necessarily equal as they depend on a choice of composition. This means we have to rethink our approach.

Fortunately, Section 10 offers a reasonable solution: Using fibrations instead of functors. So, instead of

(1) starting with functors,
(2) introducing the Grothendieck construction,
(3) introducing fibrations
(4) Translating the Yoneda lemma to the setting of fibrations,

we will start with a notion of fibration for Segal spaces and directly observe a Yoneda lemma for them.

Remark 11.1. It is in fact possible to recover the Grothendieck construction and the functorial perspective in the higher categorical setting, but this requires more advanced methods (concretely a higher categorical version of the Grothendieck construction) and will not be our focus here.

The relevant fibration has been developed with various different names. We now first define left fibrations (Definition 11.3) and then prove they can be characterized in several alternative ways: Lemmas 11.10 to 11.13. In the next section we will then study the Yoneda lemma for left fibrations.

How can we define a higher categorical version of a discrete Grothendieck opfibration? Recall that opfibrations are characterized by *unique* lifting condition, and we learned in Section 6 that the homotopical analogue to uniqueness is *contractibility*. We hence need to articulate a "contractible lifting condition". Here we use Lemma 10.18 and the following helpful analogue to a pullback square:

Definition 11.2. A square of Kan complexes with p and q Kan fibrations

$$\begin{tikzcd} W \arrow[r] \arrow[d, two heads, "q"'] & Y \arrow[d, two heads, "p"] \\ Z \arrow[r] & X \end{tikzcd}$$

is called a *homotopy pullback square* if the comparison map $W \to Z \underset{X}{\times} Y$ is an equivalence of Kan complexes.

This motivates the following definition:

Definition 11.3. A map of simplicial spaces $p : L \to X$ is called a *left fibration* if it is a Reedy fibration such that the following square is a homotopy pullback square for all $n \geq 0$

$$\begin{tikzcd} L_n \arrow[r, "0^*"] \arrow[d, "p_n"] & L_0 \arrow[d, "p_0"] \\ X_n \arrow[r, "0^*"] & X_0 \end{tikzcd}$$

Let us make some basic observations regarding left fibrations.

Lemma 11.4. *Let the following diagram be a pullback square of simplicial spaces*

$$\begin{tikzcd} W \arrow[r] \arrow[d, two heads, "q"'] & Y \arrow[d, two heads, "p"] \\ Z \arrow[r] & X \end{tikzcd}.$$

If p is a left fibration, then q is also a left fibration.

Proof. Exercise 11.2 □

Lemma 11.5. *Let $f\colon Y \to Z$ and $g\colon Z \to X$ be maps of simplicial spaces. If f and g are left fibrations, then so is the composition $g \circ f\colon Y \to X$.*

Proof. Exercise 11.3 □

For this first example we recall from Example 1.19 that Δ^0 is the simplicial space of the form

$$\{0\} \underset{d_1 \;\cong}{\overset{d_0 \;\cong}{\leftleftarrows}} \{00\} \underset{d_2 \;\cong}{\overset{d_0 \;\cong}{\leftleftarrows}} \{000\} \cdots .$$

We now have the following analogues to Exercise 10.13.

Example 11.6. A map $p : L \to \Delta^0$ is a left fibration if and only if L is homotopically constant, meaning the map $L_n \to L_0$ is an equivalence of Kan complexes for all $n \geq 0$.

For this next example we recall from Example 1.22 that Δ^1 is the simplicial space of the form

$$\{0,1\} \underset{d_1}{\overset{d_0}{\leftleftarrows}} \{00,01,11\} \underset{d_2 \;\cong}{\overset{d_0 \;\cong}{\leftleftarrows}} \{000,001,011,111\} \cdots .$$

This means every map of simplicial space $p : L \to \Delta^1$ can be depicted as a diagram of the form

$$L_{/0} \coprod L_{/1} \underset{d_1}{\overset{d_0}{\leftleftarrows}} L_{/00} \coprod L_{/01} \coprod L_{/11} \underset{d_2 \;\cong}{\overset{d_0 \;\cong}{\leftleftarrows}} L_{/000} \coprod L_{/001} \coprod L_{/011} \coprod L_{/111} \cdots .$$

Here, by the simplicial identities:

- $0^*\colon \; : L_{/00} \to L_{/0}, 0^*\colon \; : L_{/01} \to L_{/0}, 0^*\colon \; : L_{/11} \to L_{/1}$
- $1^*\colon \; : L_{/00} \to L_{/0}, 1^*\colon \; : L_{/01} \to L_{/1}, 1^*\colon \; : L_{/11} \to L_{/1}$

Remark 11.7. The notation $L_{/0}, L_{/01}, ...$ here is non-standard and is meant to be a notational help.

We now have the following analogue to Exercise 10.15.

Example 11.8. A Reedy fibration $p : L \to \Delta^1$ is a left fibration if all maps

$0^*\colon L_{/00} \to L_{/0}, 0^*\colon L_{/01} \to L_{/0}, 0^*\colon L_{/11} \to L_{/1}$
$0^*\colon L_{/000} \to L_{/0}, 0^*\colon L_{/001} \to L_{/0}, 0^*\colon L_{/011} \to L_{/01}, 0^*\colon L_{/111} \to L_{/1}$
...

are equivalences of Kan complexes.

Unwinding the structure, this means the data of a left fibration can be reduced to a zig-zag of equivalences of Kan complexes

$$L_{/0} \xleftarrow{\simeq} L_{/01} \xrightarrow{1^*} L_{/1}$$

So, up to equivalence, a left fibration over Δ^1 is just a map of Kan complexes $L_{/0} \to L_{/1}$. However, that is precisely the data of a functor out of Δ^1.

Similar to the contrast between fibrations and opfibrations, there is a contravariant analogue to left fibrations, called right fibrations.

Definition 11.9. A map of simplicial spaces $p : R \to X$ is called a *right fibration* if it is a Reedy fibration such that the following square is a homotopy pullback square for all $n \geq 0$

$$\begin{tikzcd} R_n \arrow[r, "n^*"] \arrow[d, "p_n"] & R_0 \arrow[d, "p_0"] \\ X_n \arrow[r, "n^*"] & X_0 \end{tikzcd}$$

Let us now proceed to several alternative characterizations of left fibrations.

Lemma 11.10. *Let $p : Y \to X$ be a Reedy fibration. Then p is a left fibration if and only if for each $n \geq 0$, the map*

$$(p_n, 0^*) : Y_n \twoheadrightarrow X_n \underset{X_0}{\times} Y_0$$

is a trivial Kan fibration.

Proof. Exercise 11.5 □

Next we give an alternative, inductive characterization of left fibrations.

Lemma 11.11. *Let $p : Y \to X$ be a Reedy fibration. Then p is a left fibration if and only if the commutative square*

$$\begin{tikzcd} Y_n \arrow[r, "(0,\ldots,n-1)^*"] \arrow[d, "p_n"] & Y_{n-1} \arrow[d, "p_{n-1}"] \\ X_n \arrow[r, "(0,\ldots,n-1)^*"] & X_{n-1} \end{tikzcd}$$

is a homotopy pullback square for all $n \geq 1$.

Proof. Exercise 11.6 □

Lemma 11.12. *Let $p : L \to X$ be a Reedy fibration. Then the following are equivalent:*

(1) p is a left fibration.
(2) For every map $\Delta^n \times \mathbb{D}^l \to X$ the pullback $L \times_X (\Delta^n \times \mathbb{D}^l) \to \Delta^n \times \mathbb{D}^l$ is a left fibration.
(3) For every map $\Delta^n \to X$ the pullback $L \times_X \Delta^n \to \Delta^n$ is a left fibration.

Proof. Exercise 11.7 □

Lemma 11.13. *Let $p : L \to X$ be a Reedy fibration. Then the following are equivalent:*

(1) p is a left fibration.
(2) For all $n \geq 0$

$$\begin{tikzcd} \mathrm{Map}_{s\mathcal{S}}(\Delta^n \times \Delta^1, L) \arrow[r] \arrow[d] & \mathrm{Map}_{s\mathcal{S}}(\Delta^n \times \Delta^1, X) \arrow[d] \\ \mathrm{Map}_{s\mathcal{S}}(\Delta^n, L) \arrow[r] & \mathrm{Map}_{s\mathcal{S}}(\Delta^n, X) \end{tikzcd}$$

is a homotopy pullback square.

(3) The map

$$L^{\Delta^1} \to L \underset{X}{\times} X^{\Delta^1}$$

is an equivalence of simplicial spaces.

Proof. Exercise 11.8 □

We can finally move on to the Yoneda lemma for left fibrations. The first step is to define the analogue of a representable functor in the setting of left fibrations. We saw in Example 10.26 that the Grothendieck construction of a representable functor is the slice category. This motivates the following definition:

Definition 11.14. Let X be a Segal space and x an object therein. Then the *representable left fibration* $X_{x/} \to X$ is defined as the pullback

$$\begin{tikzcd} X_{x/} \arrow[r] \arrow[d, "p"] & X^{\Delta^1} \arrow[d, "(0^*,1^*)"] \\ X \arrow[r, "(x,\mathrm{id}_X)"] & X \times X \end{tikzcd}.$$

We now have the following major result regarding $p \colon X_{x/} \to X$.

Proposition 11.15. *Let X be a Segal space and x an object therein. Then the map $p : X_{x/} \to X$ is a left fibration.*

Before we can state the main result, we need one last definition.

Definition 11.16. Let X be a simplicial space, and $p\colon Y \to X, q : Z \to X$ be to morphisms. Then the *relative mapping space* $\mathrm{Map}_{/X}(Y, Z)$ is defined as the pullback

$$\begin{tikzcd} \mathrm{Map}_{/X}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Map}(Y,Z) \arrow[d, "q_*"] \\ \mathbb{D}^0 \arrow[r, "p"] & \mathrm{Map}(Y,X) \end{tikzcd} .$$

Remark 11.17. The definition of $\mathrm{Map}_{/X}(Y, Z)$ depends on the choice of maps p, q, but we will suppress this in the notation.

We can now state the analogue of Lemma 10.27 in the setting of left fibrations.

Theorem 11.18. *Let X be a Segal space and x an object therein. Moreover, let $L \to X$ be a left fibration. Then the map*

$$\begin{array}{ccc} \mathrm{Map}_{/X}(X_{x/}, L) & \xrightarrow{\simeq} & \mathrm{Map}(\Delta^0, L) = \{x\} \underset{X}{\times} L \\ [f\colon X_{x/} \to L] & \longmapsto & f(\mathrm{id}_x) \end{array}$$

is an equivalence of Kan complexes.

Exercises.

Exercise 11.1. Explain how the pullback condition in Definition 11.3 is indeed a homotopically unique lifting condition.

Exercise 11.2. Prove Lemma 11.4: Left fibrations are stable under pullback.

Exercise 11.3. Prove Lemma 11.5: Left fibrations are stable under composition.

Exercise 11.4. Use an argument similar to Example 11.8 to characterized a left fibration $p : L \to \Delta^n$ as a diagram of Kan complexes

$$L_{/0} \xleftarrow{\simeq} L_{/01} \xrightarrow{1^*} L_{/1} \xleftarrow{\simeq} L_{/12} \xrightarrow{2^*} L_{/2} \xleftarrow{\simeq} \dots \xleftarrow{\simeq} L_{/(n-1)n} \xrightarrow{n^*} L_{/n}$$

Exercise 11.5. Prove Lemma 11.10: p is a left fibration if and only if for each $n \geq 0$, the map $Y_n \twoheadrightarrow X_n \underset{X_0}{\times} Y_0$ is a trivial Kan fibration.

Exercise 11.6. Prove Lemma 11.11: p is a left fibration if and only if the commutative square

$$\begin{tikzcd} Y_n \arrow[r, "(0,\dots,n-1)^*"] \arrow[d, "p_n"] & Y_{n-1} \arrow[d, "p_{n-1}"] \\ X_n \arrow[r, "(0,\dots,n-1)^*"] & X_{n-1} \end{tikzcd}$$

is a homotopy pullback square for all $n \geq 1$.

Exercise 11.7. Prove Lemma 11.12: Show that being left fibration admits alternative characterizations in terms of pullbacks along representables.

Exercise 11.8. Prove Lemma 11.13: Show that being left fibration admits alternative characterizations in terms of mapping spaces and exponentials.

Exercise 11.9. Let $0\colon \Delta^0 \to \Delta^1$ and take for granted it is a Reedy fibration. Prove or give a counterexample:

(1) 0 is a left fibration.

(2) 0 is a right fibration.

Exercise 11.10. Let $1\colon \Delta^0 \to \Delta^1$ and take for granted it is a Reedy fibration. Prove or give a counterexample:

(1) 1 is a left fibration.
(2) 1 is a right fibration.

Exercise 11.11. Let $P\colon \mathcal{D} \to \mathcal{C}$ be a functor. Prove the following are equivalent

(1) P is a discrete Grothendieck fibration.
(2) discNP is a left fibration.

Exercise 11.12. Let W be a Segal space. Prove that a map $p\colon L \to W$ is a left fibration if and only if the square

$$\begin{tikzcd} L_1 \arrow[r, "d_1"] \arrow[d, "p_1"] & L_0 \arrow[d, "p_0"] \\ W_1 \arrow[r, "d_1"] & W_0 \end{tikzcd}$$

is a homotopy pullback square.

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