Elementary ∞-Topos Theory: Constructing Coproducts in locally Cartesian closed ∞-Categories

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January 20th, 2022

What am I doing here?

Slogan

I am a homotopy theorist who wants a working foundation for homotopical mathematics!

How to do Foundations in non-homotopical math

Non-Homotopists have so many choices:

1. Set theories
2. Type theories
3. Elementary toposes

You can choose your axioms and then do your math.

Menu of Axioms

1. If we have a weakly inaccessible cardinal then ...
2. If the law of excluded middle holds then ...
3. If the axiom of choice holds then ...
4. If the continuum hypothesis holds then ...

Homotopy type theory as a Foundation

Homotopy type theory and univalent foundations is one elegant example of such possible foundation! It gives us model-independence of results such as

1. Loop space of the circle is the free group on one generator
2. Blakers-Massey
3. Hurewicz theorem
4. ...

Elementary ∞-Topos Theory

Another possible foundation,

1. Closely (expected to be) related to type theory
2. Much less understood!
3. I want to know more about them!

History

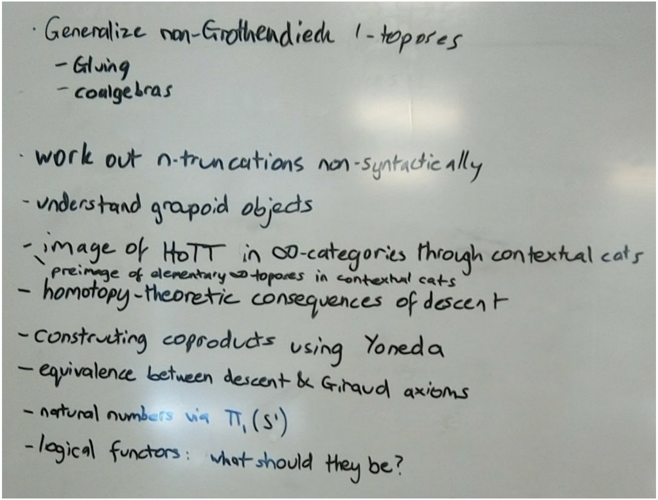

Figure: Chris Kapulkin on 6/5/2017 in Snowbird at MRC.

We have some answers

1. Examples: *Filter Quotients and Non-Presentable $(\infty, 1)$-Toposes*
2. N-Truncations: *An Elementary Approach to Truncations*
3. Coproducts: *Constructing Coproducts in locally Cartesian closed ∞-Categories*
4. Natural number objects: *Every Elementary Higher Topos has a Natural Number Object*
5. Logical functors: *A Theory of Elementary Higher Toposes*

The work on finite coproducts started with Jonas Frey on that date!

Let's Start: ∞-Categories

Our goal is to construct colimits in certain ∞-categories. What are ∞-categories?

1. If you know: **quasi-categories** (but any biequivalent ∞-cosmos works).
2. If you don't know: A type of category weakly enriched over spaces, where all standard notions of category theory exist.

What's up with topos theory and colimits?

Why do we expect to be able to construct coproducts in certain ∞-categories?

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Why do we expect to be able to construct coproducts in certain ∞-categories?

Answer

Because of existing results in elementary topos theory!

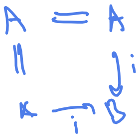
Subobject classifier

$\mathcal{C}$ a finitely complete ∞-category. A subobject classifier is ...

- An object Ω
- A mono $t : 1 \to \Omega$

such that pulling back t gives us a bijection

$$\mathrm{Sub}(-) \cong \mathrm{Map}(-, \Omega).$$



Elementary Toposes and Finite Colimits

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However, we have:

Theorem (Paré, Mikkelsen)

Every **locally Cartesian closed** 1*-category with* **subobject classifier** *has finite colimits.*

So, the assumption of finite colimits in an elementary topos is redundant.

The ∞-Version

Can we prove (or disprove) the ∞-categorical analogue?

Conjecture

Every locally Cartesian closed ∞-category with subobject classifier has finite colimits.

How does the 1-Categorical Proof Work?

The proof by Paré is a one package deal. It consists of showing that the functor

$$\Omega^{(-)} : \mathcal{E}^{op} \to \mathcal{E}$$

is monadic and so $\mathcal{E}^{op}$ has finite limits (as $\mathcal{E}$ has them) and so $\mathcal{E}$ has finite colimits.

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Not even monadic for spaces! In fact not even conservative for 1-types:

$$\Omega^{S^1} \cong \{0,1\}^{S^1} \cong \{0,1\}$$

How does the 1-Categorical Proof Work? (Second Try)

The proof attributed to Mikkelsen is more adhoc:

1. First we realize $\mathrm{Hom}(1, \Omega^A) \cong \mathrm{Sub}(A)$ is a Heyting algebra!
2. The initial object is the minimal subobject
3. The coproduct of A and B is given as the join inside $\Omega^A \times \Omega^B$.
4. Coequalizers are constructed analogously via equivalence relations.

We want to generalize these steps to the ∞-setting!

Subobjects

First of all even in an ∞-category $\mathrm{Sub}(-)$ is a partially ordered **set** (0-type). So using similar ideas we have:

Theorem (Frey - R.)

Let $\mathcal{C}$ be a locally Cartesian closed ∞-category with subobject classifier. Then $\mathrm{Sub}(A)$ has finite joins.

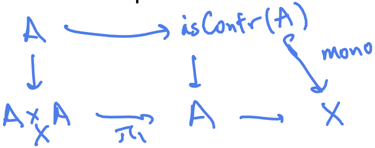
Initial Objects I

The argument for initial objects in the ∞-setting is roughly similar

1. Ω is a Heyting algebra and so has an initial object I. This means $\mathrm{Sub}(I)$ is trivial.
2. **Homotopy type theory fact:** We have a functor

$$\mathrm{isContr} : \mathcal{C}_{/X} \to \mathrm{Sub}(X),$$

which takes a map to the maximal subobject if and only if it is an equivalence.



Initial Objects II

3. By (2), for an object X, $\mathrm{Sub}(X)$ is trivial if and only if $\mathcal{C}_{/X}$ is trivial. So by (1) all of $\mathcal{C}_{/I}$ is trivial.
4. Finally, X^I is the pushforward of the equivalence $X \times I \to I$ along $I \to 1$ and so is terminal, meaning

$$\mathrm{Map}(I, X) \simeq \mathrm{Map}(1, X^I)$$

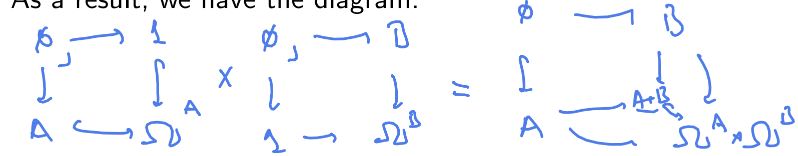
is a contractible space.

Disjoint Subobjects in 1-Categories

Why does the Mikkelsen coproduct argument work? In an elementary topos we have

1. $A \to \Omega^A$ is a subobject!
2. $f : 1 \to \Omega^A$ is a disjoint point!

As a result, we have the diagram:



Disjoint Subobjects in ∞-Categories

This argument generalizes:

Lemma

Let A, B be two objects in a locally Cartesian closed ∞-category with subobject classifier and assume an object C exists such that $A \hookrightarrow C \hookleftarrow B$. Then the join in $\mathrm{Sub}(C)$ is the coproduct of A and B.

$\mathcal{E}$ disjoint

Proof 1

Assume $\underline{A+B} \simeq C = \underline{1}$

$$\mathrm{Map}(1, X) \to \mathrm{Map}(A, X) \times \mathrm{Map}(B, X)$$

$$\begin{array}{ccc} \underset{\text{contractible}}{P} & \longrightarrow & X \\ \downarrow \simeq & & \downarrow \\ 1 & \longrightarrow & X^A \times X^B \end{array}$$

Proof 2

$\Leftarrow$ is $\mathrm{Contr}(P) = 1$ in $\mathrm{Sub}(1)$

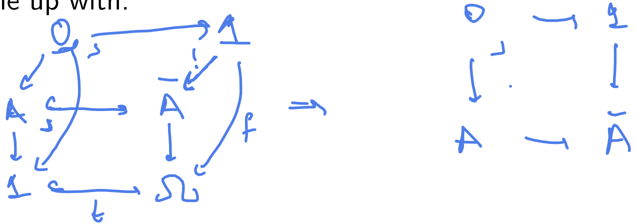
$$\begin{cases} A^* \text{ is } \mathrm{Contr}(P) = A \\ B^* \text{ is } \mathrm{Contr}(P) = B \end{cases}$$

$$A\left(\begin{array}{ccc} P & \to & X \\ \downarrow & & \downarrow \\ 1 & \to & X^A \times X^B \end{array}\right) = \begin{array}{ccc} A^*P & \to & A^*X \\ \simeq\downarrow & & \downarrow = \\ A & \to & A^*X \end{array}$$

$$\begin{cases} A^*X^A = AX \\ A^*X^B = 1 \end{cases}$$

How to find the right Superobject?

All that is left to construct coproducts is to show every object has a superobject with disjoint point! $A \to \Omega^A$ is not mono (unless A is 0-truncated) so that doesn't work. Here is the idea that we came up with:



How about coequalizers?

Coequalizers fail.

Example

The ∞-category of eventually truncated spaces is locally Cartesian closed and has a subobject classifier $\{0, 1\}$, but the suspension of (for example) S^1 does not exist.

Example (Anel)

Truncated coherent spaces are locally Cartesian closed and have a subobject classifier $\{0, 1\}$, but also do not have coequalizers, as the suspension of S^0 does not exist.

The corrected conjecture

Theorem (Frey - R.)

Let $\mathcal{C}$ be a locally Cartesian closed ∞-category with subobject classifier. Then it has a strict initial object and disjoint universal finite coproducts. However, there are examples where it does not have coequalizers.

How do we get finite colimits?

The next naive conjecture goes along the following lines:

Conjecture

Let $\mathcal{E}$ be a locally Cartesian closed ∞-category with subobject classifier and sufficient univalent universes. Then $\mathcal{E}$ has finite colimits.

The conditions are (more or less) the current definitions of an elementary ∞-topos, so that would match well with the results in the elementary topos world, justifying the additional assumption.

The End!

Questions?

- **Source:** Constructing Coproducts in locally Cartesian closed ∞-Categories, arXiv:2108.11304
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Thank You!