LECTURE NOTES ON HIGHER CATEGORIES

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Abstract. These are lecture notes for a course on higher categories in Spring 2020 at École Polytechnique Fédérale de Lausanne. The goal is to focus on certain aspects that are usually undercovered in the ∞-category world. In particular, the fact that an ∞-category is a concrete object: a simplicial set, a simplicial space, a simplicial category, ... and there are various ways to move between these notion.

Contents

What is $(\infty, 1)$-Category Theory about?

The goal of $(\infty, 1)$-category theory is to study categories which have some sort of input from homotopy theory. This can happen in various forms:

(1) Many categories (such as Kan complexes themselves) are enriched over Kan complexes and we use the structure of the mapping space to study "homotopic morphisms".
(2) Many categories come with a notion of *equivalence* in addition to their standard notion of isomorphism. Again, Kan complexes and homotopy equivalences are an example. But

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we have other examples less obviously topological, such as (bounded) chain complexes and quasi-isomorphisms.

So, the goal is to give a definition of a generalization category that incorporates such a data. The collection of definitions that satisfy these conditions are known as $(\infty, 1)$-categories. Thus $(\infty, 1)$-category itself is not really a definition but rather a guiding idea that can help us determine what definition satisfies the desired condition.

Remark 0.1. The notation $(\infty, 1)$ comes from the idea that we have an infinite level of morphisms (the ∞), where all morphisms beyond level 1 are invertible and goes back to a time when $(\infty, 1)$-category theory was thought of as a generalization of strict n-category theory.

A Zoo of $(\infty, 1)$-Categories

Let us now see the various models of $(\infty, 1)$-categories that appear in the literature.

1.1 Relative Categories. The goal is thus is to come up with a working definition of a category that has the data of an $(\infty, 1)$-category. Let us start by making a first effort.

Definition 1.1. A relative category is a pair $(\mathcal{C}, \mathcal{W})$ of a 1-category $\mathcal{C}$ along with a subcategory $\mathcal{W}$ that contains all the objects.

Remark 1.2. Sometimes we make the additional assumption that $\mathcal{W}$ contains all the isomorphisms.

The way we should think about relative categories is that the subcategory $\mathcal{W}$ exactly gives us the desired weak equivalences. Thus we can easily extract following data from a relative category:

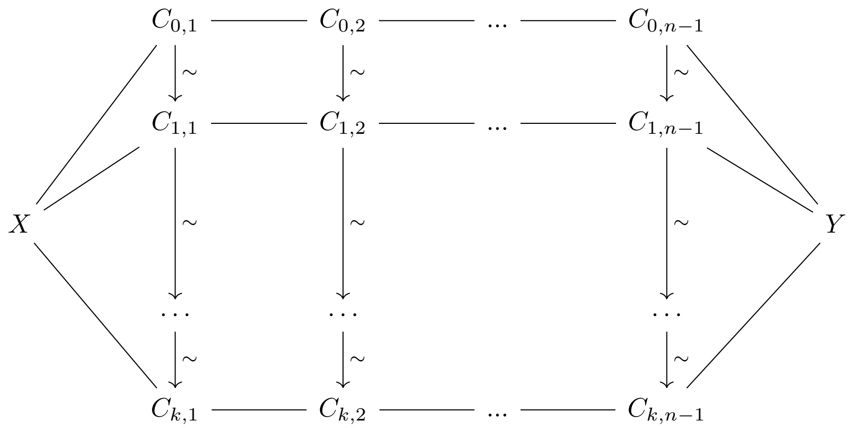
(1) Objects are just object of $\mathcal{C}$.
(2) Morphisms are just morphisms of $\mathcal{C}$
(3) Weak equivalences are just morphisms in $\mathcal{W}$.

However, past the obvious observations this definition is very hard to use for many practical things. First of all we don't actually know when two are objects are "equivalent". That is because two objects are equivalent if they are connected by a zigzag of morphisms in $\mathcal{W}$, which is hard to study. Even more importantly, it is not clear how to study limits and colimits and other categorical concepts in this setting.

Remark 1.3. Relative categories usually arise as *model categories*. However, there the existence of fibrations and cofibrations leads to actual meaningful categorical constructions, such as *Quillen adjunctions* and *homotopy (co)limits*.

The problem with relative categories is that it is very minimalistic. An effective definition would have some data that includes the data of the subcategory $\mathcal{W}$, but also some additional data that is determined by those weak equivalences, but can help us do some effective category theory.

A first key insight was given by Dwyer and Kan in their paper simplicial localizations of categories. In that paper they prove that out of every relative category (which is not the original terminology) we can construct a simplicially enriched category. For two objects X, Y a k-cell is a diagram of the form



where all the vertical maps are in the relative category $\mathcal{W}$, but that also satisfies several other conditions (for more details see [DK80a] and in particular [DK80b, 2.1]).

So, every relative category very readily gives us a simplicially enriched category. And in particular we easily get a Kan enriched category by simply Kan fibrantly replacing those mapping simplicial sets.

1.2 Kan-Enriched Categories. Let us now try to repeat our studies, but this time we focus on Kan enriched categories.

Definition 1.4. A Kan enriched category is a 1-category enriched over the category of Kan complexes (which is a Cartesian closed category).

Kan enriched categories are nice. They are just a special case of an enriched category. Thus we can use all the tools we have for enriched categories. In particular, this definition immediately comes with appropriate notions of adjunction, limit, colimits, ... The constructions are very analogous to the constructions for set-enriched categories and so we won't discuss them any further. For a good reference on enriched category theory, see for example [Ri14].

The only thing we do want to mention is: How do we recover our "weak equivalences" from a Kan enriched category? Here we have to use our intuition from homotopy theory:

Definition 1.5. Let $\mathcal{C}$ be a Kan enriched category. Then two morphisms $f, g : X \to Y$ are *homotopic*, denoted $f \simeq g$, if they are in the same path-component of the Kan complex $Map_{\mathcal{C}}(X, Y)$.

Definition 1.6. A morphism $f : X \to Y$ is an *equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

This immediately gives us an "inverse" to the argument in the last section.

Theorem 1.7. *Let $\mathcal{C}$ be a Kan enriched category. Let $\mathcal{C}_0$ be the category 0-cells and W_0, the subcategory of equivalences in $\mathcal{C}_0$. Then $(\mathcal{C}_0, \mathcal{W}_0)$ is a relative category.*

This only relies on the basic observation that an identity map is always an equivalence.

Thus a Kan enriched category has a notion of weak equivalence, but is also Kan enriched and has a very developed theory of categories. Why keep going? Why not just stop here?

Here is a basic observation: Whenever, we have a set S we can upgrade it to a category $\mathcal{C}(S)$ where the objects is just the set S and all morphisms are identities. We would expect a direct homotopical generalization of this fact: Namely, for a given Kan complex X, we want a Kan enriched category $\mathcal{C}(X)$ defined as:

(1) Objects is the set X_0.
(2) For two objects x, y in X_0, the mapping space should be
$$Map_{\mathcal{C}(X)}(x,y) = \mathcal{P}\mathrm{ath}(x,y)$$
where $\mathcal{P}\mathrm{ath}(x,y)$ is the path space between x and y.

However, this doesn't work. Let us focus on the specific case where $X_0 = \{x\}$, a single point. In that case
$$\mathcal{P}\mathrm{ath}(x,x) = \Omega_x X$$
The Kan complex $\Omega_x X$ is not a monoid! Rather it is an A_∞-space. So there is a clear mismatch between topology and category theory. Topology requires us to study homotopical algebra, whereas category theory needs strict algebras. So, we either have to strictify all these necessary spaces (which can be done) or we have to unstrictify our notion of category.

However, even if we do so, it is a very unnatural for a homotopy theorist to think about a space as a collection of points and choice of path spaces. This annoyance is not just theoretical, but actually shows up whenever objects and morphisms interact in our Kan enriched category.

Example 1.8. Let I be a the free isomorphism (a Kan fibrant replacement of Δ^1) and let ΣI be the category with two objects 0 and 1 such that
$$Map(0,0) = \{id_0\}$$
$$Map(0,1) = I$$
$$Map(1,0) = \emptyset$$
$$Map(1,1) = \{id_1\}$$
Let us denote the zero cells in $Map(0,1)$ as f, g. We want to define the slice category $(\Sigma I)_{/1}$. We would expect the objects to be the 0-cell morphisms, which are
$$id_1, f, g$$
What are the morphisms? Naively, we expect to have following pullback
$$\begin{array}{ccc} Map_{(\Sigma I)_{/1}}(f,g) & \longrightarrow & Map_{\Sigma I}(0,0) \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle g_*} \\ * & \xrightarrow[\{f\}]{} & Map_{\Sigma I}(0,1) \end{array}$$
Using the definitions given above we get the pullback

$$\begin{array}{ccc} Map_{(\Sigma I)_{/1}}(f,g) & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow {\scriptstyle \{g\}} \\ * & \xrightarrow[\{f\}]{} & I \end{array}$$

but this pullback is empty! This does not match our intuition. We would expect there to be a morphism between the objects f and g that corresponds to the path I.

It is clear how to fix this from a homotopical perspective. We need a notion of *homotopy pullback*. But this would throw everything else out of balance. If we replace limits with homotopy limits, then it is for example not clear how we can recover the composition of the mapping spaces. This is analogous to following situation. Let us assume we have a space X, which is given as a set of points and its path spaces. How can we find the path fibration?

All these problems point in a single direction: We need a more global approach to category theory! We need a notion of $(\infty, 1)$-category in which objects are morphisms are put on the same level, which is already the case in homotopy theory and would allow us to study morphisms as objects whenever we desire. In order to do that we will change our perspective on 1-categories.

1.3 Nerves of Categories. Categories are usually defined as a set of objects and hom sets that come with the correct composition maps satisfying the desired conditions. This definition has exactly the problem we pointed to above as it distinguishes between objects and morphisms. We need a global approach. Fortunately, this already exists.

Definition 1.9. Let

$$N : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$$

be the functor determined by the cosimplicial category

$$[0] \rightrightarrows [1] \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; [2] \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

i.e. it takes a category $\mathcal{C}$ to the simplicial set

$$Fun([0], \mathcal{C}) \leftleftarrows Fun([1], \mathcal{C}) \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; Fun([2], \mathcal{C}) \;\substack{\longleftarrow\\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

Notice in particular that $N(\mathcal{C})_0$ are the objects of $\mathcal{C}$ and $N(\mathcal{C})_1$ are the morphisms of $\mathcal{C}$. Moreover, $N(\mathcal{C})_2$ is the set of composable morphisms and the boundary map $d_1 : N(\mathcal{C})_2 \to N(\mathcal{C})_1$ the composition map. Here is a key property of this functor.

Theorem 1.10. *The functor N is an embedding.*

Proof. We need to prove that for two categories $\mathcal{C}, \mathcal{D}$ the map

$$Fun(\mathcal{C}, \mathcal{D}) \to Hom(N\mathcal{C}, N\mathcal{D})$$

is a bijection. For that we need an inverse. Let $\alpha : N\mathcal{C} \to N\mathcal{D}$. Then we can define

$$F_\alpha : \mathcal{C} \to \mathcal{D}$$

as a functor that is defined on objects as $\alpha_0 : N\mathcal{C}_0 \to N\mathcal{D}_0$ and on morphisms as $\alpha_1 : N\mathcal{C}_1 \to N\mathcal{D}_1$. It remains to see that F_α respects composition. But this follows from the commutative diagram

$$\begin{array}{ccc} N\mathcal{C}_2 & \xrightarrow{d_1} & N\mathcal{C}_1 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} \\ N\mathcal{D}_2 & \xrightarrow{d_1} & N\mathcal{D}_1 \end{array}$$

□

Actually we can do even more, namely we can characterize the image of the nerve functor:

Theorem 1.11. *A simplicial set S is in the image of N if and only if it satisfies the Segal condition: For $n \geq 2$, there is a bijection*

$$S_n \xrightarrow{\cong} S_1 \underset{S_0}{\times} \ldots \underset{S_0}{\times} S_1$$

Proof. We need to find a category $\mathcal{C}$ such that $N(\mathcal{C}) = S$, however the condition exactly tells us how to define everything. Namely, the set of objects is S_0, the set of morphisms S_1 and the composition is the map

$$S_1 \underset{S_0}{\times} S_1 \cong S_2 \xrightarrow{\cong} S_1$$

The higher Segal conditions imply that this composition is associative. □

The upshot: Categories live inside simplicial sets as the simplicial sets that satisfy the Segal condition. This suggests that our generalization of a category i.e $(\infty, 1)$-category should be in the world of simplicial objects rather than categorical objects.

1.4 The Simplicial Approach. How can we use simplicial objects to define a working notion of $(\infty, 1)$-categories? We will focus on two ways to achieve this.

Here is our first approach: Recall that a Kan complex is a simplicial set K that satisfies the lifting condition with respect to all horns

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

On the other hand, the Segal conditions means that we have **unique** lifts

$$\begin{array}{ccc} Sp^n & \longrightarrow & S \\ \downarrow & \overset{\exists!}{\nearrow} & \\ \Delta^n & & \end{array}$$

where Sp^n is the spine of Δ^n, meaning sub-complex $\Delta^1 \coprod_{\Delta^0} .... \coprod_{\Delta^0} \Delta^1$.

If we combine these two we could make following non-unique lifting assumption:

$$\begin{array}{ccc} Sp^n & \longrightarrow & S \\ \downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

However, this is wrong! In particular it fails to give us a good notion of associativity.

Example 1.12. Let S be the simplicial set

$$S = \Delta^3 \coprod_{\partial\Delta^2} \Delta^2$$

where the map $\partial\Delta^2 \to \Delta^3$ is given by the subset 123. Notice it satisfies the lifting condition given above, as Δ^3 satisfies it (being $N([3])$ and we did not modify the 0 and 1-cells. If such a definition gave us a functioning definition of a category, then $0, 1, 2, 3$ should play the role of objects, and we should have a mapping space $Map(0, 3)$. This mapping space should have 4 0-cells, corresponding to the four maps from 0 to 3 and then we have various homotopies between them. The homotopies should then in particular imply some homotopical notion of associativity. However, the mapping space concretely has the following form:

$$\begin{array}{ccc} 0123 & \rightrightarrows & 023 \\ \downarrow & & \downarrow \\ 013 & \longrightarrow & 03 \end{array}$$

if we choose the upper path and then down $0123 \to 023 \to 03$, then there is no witness for its associativity we cannot find a homotopy to the path $0123 \to 013 \to 03$.

We need to think a little bit more general. The key is to recognize that the Segal condition on a simplicial set can be generalized to an equivalent lifting condition that is a better fit for our situation.

Lemma 1.13 ([Re17, Proposition 5.9])**.** *A simplicial set S satisfies the Segal condition if and only if we have* **unique** *lifts along all inner horns*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \downarrow & \overset{\exists !}{\nearrow} & \\ \Delta^n & & \end{array}$$

We have already observed that our notion of $(\infty, 1)$-category should include categories and Kan complexes, which means it should satisfy the common of those two conditions (an arrow indicates more restrictive conditions):

$$\begin{array}{ccc} (\infty,1)-\text{categories} = \text{lift along inner horns} & \longrightarrow & \text{categories} = \text{unique lift along inner horns} \\ \downarrow & & \downarrow \\ \text{Kan complexes} = \text{lift along horns} & \longrightarrow & \text{Groupoids} = \text{unique lift along inner horns} \end{array}$$

This actually gives us a working definition.

Definition 1.14. A *quasi-category* is a simplicial set that satisfies the lifting condition along inner horns.

Before we actually say anything meaningful about quasi-categories let us try a different method.

As we observed before, we need two kinds of conditions. In the previous step we applied the smallest common denominator to get a working definition. However, we could have a different course of action. Namely using two different simplicial axes and giving each the necessary condition.

For that we need a review of simplicial spaces.

Definition 1.15. A simplicial space is a bisimplicial set, meaning a functor

$$X : \Delta^{op} \times \Delta^{op} \to \mathcal{S}\text{et}$$

In order to study it we need generators. Here we use notation introduced by Charles Rezk. We define

$$\Delta^l_{km} = Hom([m], [l])$$
$$F(n)_{km} = Hom([k], [n])$$

As a convention, we think of $F(n)$ as the horizontal generator and Δ^l as the vertical generator. The category of simplicial spaces is generated by the products $F(n) \times \Delta^l$. We can now use $F(n)$ and Δ^l to define our desired simplicial spaces.

Simplicial spaces are enriched over themselves, but also enriched over simplicial sets. Concretely:

$$Map(X, Y)_l = Hom(X \times \Delta^l, Y)$$
$$(Y^X)_{kl} = Hom(X \times F(k) \times \Delta^l, Y)$$

and so in particular

$$(Y^X)_0 = Map(X, Y)$$

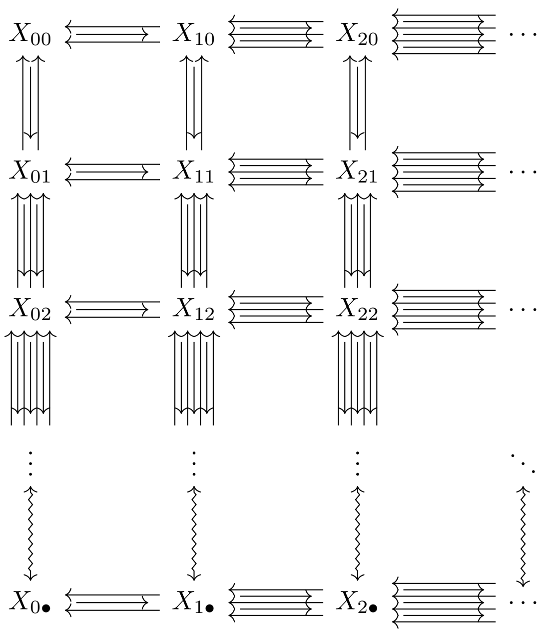
A key observation of this enrichment and representation is that:

$$Map(F(n), X)_l = Hom(F(n) \times \Delta^l, X) \cong X_{nl}$$

and so

$$Map(F(n), X) \cong X_n$$

We can depict this information in the following diagram:



We will use this enrichment and this way of depicting simplicial spaces throughout without ever mentioning things explicitly. So, in particular for given simplicial space X, X_k is a space and X_{kl} is a set.

Definition 1.16. A simplicial space X is *Reedy fibrant* if for any n the map

$$Map(F(n), X) \to Map(\partial F(n), X)$$

is a Kan fibration.

In, particular this implies that each level is a Kan complex. Thus we can say a Reedy fibrant simplicial space is vertically a Kan complex.

We now need the horizontal direction to be categorical.

Definition 1.17. A *Segal space* is a simplicial space T such that for each $n \geq 2$ the Segal map

$$T_n \xrightarrow{\simeq} T_1 \underset{T_0}{\times} ... \underset{T_0}{\times} T_1$$

is a Kan equivalence.

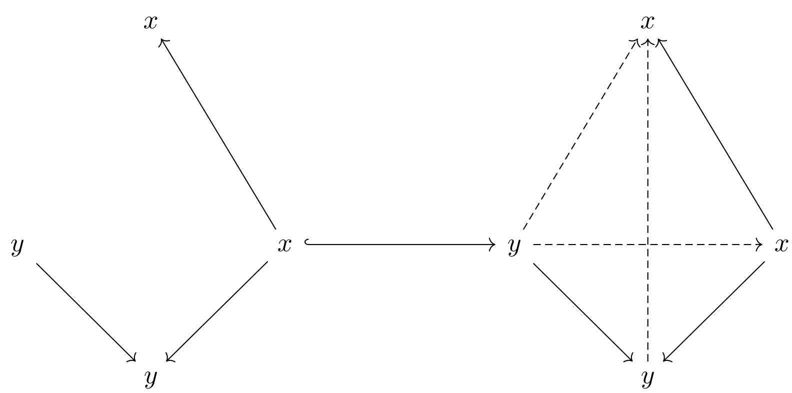
These two conditions should give us a working definition for an $(\infty, 1)$-category. However, it turns out there is a problem. The problem boils down to following observation: There is a homotopical way of being equivalent, given by the existence of paths. But there is also a categorical way of being equivalent, given by existence of inverses. These two notions don't have to agree in a general Segal space. Thus we have to ensure they agree.

This can be forced by the following condition.

Definition 1.18. A Segal space W is called *complete* if the following is a pullback square

$$\begin{array}{ccc} W_0 & \longrightarrow & W_3 \\ \downarrow & \lrcorner & \downarrow \\ W_1 & \xrightarrow[(s_0, id, s_0)]{} & W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 \end{array}$$

where the map $W_3 \to W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1$ can be expressed geometrically as

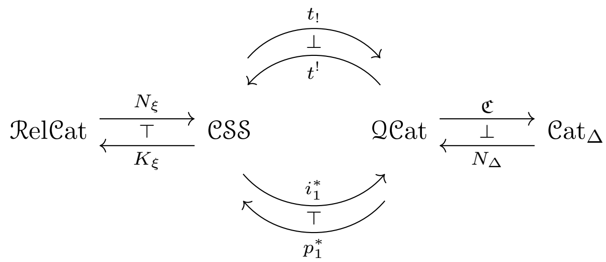


We claim that we have given two different ways of constructing $(\infty, 1)$-categories using simplicial methods: quasi-categories and complete Segal spaces.

The next step is to show that this claim is actually true. In order to give some amount of justification that this is true we want to observe the following facts:

(1) Every one of the definition of $(\infty, 1)$-category we have given comes with a model structure.
(2) All of these model structures are Quillen equivalent.

This can be summarized in the following diagram:



where

(1) The adjunction (K_ξ, N_ξ) was defined and proven by Barwick and Kan [BK12].
(2) The adjunctions $(t_!, t^!), (p_1^*, i_1^*)$ where defined and proven by Joyal and Tierney [JT07].
(3) The adjunction $(\mathfrak{C}, N_\Delta)$ was defined and proven by Lurie [Lu09].

We will not study this diagram in all details, cause it's too much time and effort, but we will focus on three particular models, namely Kan enriched categories, quasi-categories and complete Segal spaces as these are the models most common in the $(\infty, 1)$-category theory literature.

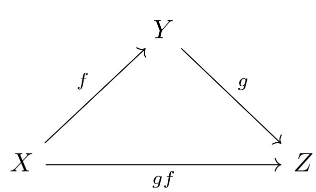
A Tale of Three Model Structures

We now want to carefully study three model structures and give some justification why these exist.

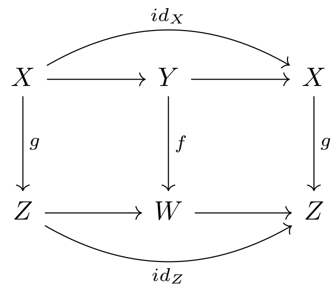
2.1 Model Categories. A *model category* is a category with additional data that allows us to do homotopical manipulation. Let us give a definition as is roughly given in [Ho98].

Definition 2.1. A model category is complete and cocomplete category $\mathcal{M}$ along with three subcategories $\mathcal{C}, \mathcal{F}, \mathcal{W}$ known as cofibrations, fibration and weak equivalences, such that they satisfy following axioms.

(1) **2-Out-Of-3**: The maps in $\mathcal{W}$ satisfy 2-out-of-3, meaning that in the diagram

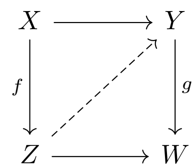


if two are in $\mathcal{W}$ then the third one is as well.
(2) **Retracts**: In the diagram



if f is in $\mathcal{C}$, $\mathcal{F}$ or $\mathcal{W}$ then g is as well.

(3) **Lifting** In the diagram



then a lift exists if $f \in \mathcal{C} \cap \mathcal{W}$ and $g \in \mathcal{F}$ or $f \in \mathcal{C}$ and $g \in \mathcal{F} \cap \mathcal{W}$.

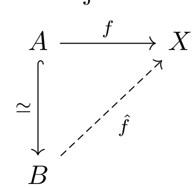
(4) **Factorization**: Every morphism can be factored into a trivial cofibration (cofibration that is a weak equivalence) followed by a fibration and a cofibration followed by a trivial fibration (fibration that is a weak equivalence).

The classes of morphisms $\mathcal{C}, \mathcal{F}$ give us some special classes of objects.

(1) An object is *fibrant* if the unique map to the final object is a fibration.
(2) An object is *cofibrant* if the unique map from the initial object is a cofibration.

Why do we care about such model categories? Several reasons:

(1) The tuple $(\mathcal{M}, \mathcal{W})$ is the data of a relative category, so a model category in particular has homotopical data.
(2) The lifting condition means that fibrant objects are determined by a lifting condition



But many objects we want to study (such as quasi-categories, complete Segal spaces and Kan enriched categories) are exactly defined by such lifting conditions.

(3) The factorization implies that we can always factor the final map $X \to *$ into a diagram

$$X \overset{\simeq}{\hookrightarrow} \hat{X} \twoheadrightarrow *$$

where the map $\hat{X} \twoheadrightarrow *$ is a fibration and so $\hat{X}$ is fibrant. Thus a model structure gives us a concrete method for constructing our desirable objects out of arbitrary objects (such as a quasi-category out of a simplicial set).

(4) There are very strong theorems that allow us to construct model categories, such as cofibrantly generated categories and Bousfield localizations.
(5) There are concrete ways of comparing two model structures and recognizing when they are equivalent via Quillen equivalences.

Example 2.2. The category of simplicial sets comes with a very important model structure, the *Kan model structure*, which has

(1) Cofibrations: Monomorphisms
(2) Fibrations: Kan fibrations
(3) Weak Equivalences: Kan Equivalences

In this example everything is cofibrant but the fibrant objects are exactly the Kan complexes, which are the central objects of study.

The goal is to use some of these techniques to construct model structures in which Kan enriched categories, quasi-categories and complete Segal spaces are the fibrant objects and then show they are equivalent. We will use following steps.

(1) First recognize what kind of data we want (in the sense of what kind of fibrant objects and weak equivalences, for example).
(2) Prove the data can be completed to a model structure via some big theorem (that we will not prove).

After we have defined the three model structures we will prove they are equivalent.

2.2 Kan Enriched Categories. The work here is due to Bergner [Be07]. Let $\mathcal{C}at_{\Delta}$ be the category of simplicially enriched categories. What we really care about are the Kan enriched categories. As, Kan complexes are the fibrant objects in the Kan model structure, we thus want a model structure on $\mathcal{C}at_{\Delta}$ such that the fibrant objects are exactly the Kan enriched categories.

What should be our fibrations? Let $F : \mathcal{D} \to \mathcal{C}$ be a map of simplically enriched categories. Clearly, we want the map on mapping spaces

$$Map_{\mathcal{D}}(x, y) \to Map_{\mathcal{C}}(Fx, Fy)$$

to be a Kan fibration. But is that enough? No, we actually need an equivalence lifting property, which leads to following definition.

Definition 2.3. A functor $F : \mathcal{C} \to \mathcal{D}$ is a fibration of simplicial categories if

(1) **Isofibration**: It has right lifting property along the map $\Sigma\Delta^0 \to \Sigma I$.

$$\begin{array}{ccc} \Sigma\Delta^0 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Sigma I & \longrightarrow & \mathcal{D} \end{array}$$

meaning that for any equivalence in $\mathcal{D}$ and choice of lift for the source, we can lift the equivalence to $\mathcal{C}$.

$$\begin{array}{ccc} \hat{c} & \overset{\simeq}{\dashrightarrow} & \hat{d} \\ \Downarrow & & \Downarrow \\ c & \xrightarrow{\simeq} & d \end{array}$$

(2) **Local Kan fibration**: For two objects c_1, c_2 in $\mathcal{C}$ the map

$$Map_{\mathcal{C}}(c_1, c_2) \to Map_{\mathcal{D}}(Fc_1, Fc_2)$$

is a Kan fibration.

Notice immediately that a category is fibrant if and only if each mapping space is a Kan complex, which was what we wanted.

What should be our notion of equivalence? Here we have to just the classical definition to the new homotopy setting.

Definition 2.4. A functor of Kan enriched categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if the following two conditions hold:

(1) **Fully Faithful**: For two objects c_1, c_2 the map of simplicial sets

$$Map_{\mathcal{C}}(c_1, c_2) \to Map_{\mathcal{D}}(Fc_1, Fc_2)$$

is a weak equivalence

(2) **Essentially Surjective**: For any object d in $\mathcal{D}$ there exists an object c in $\mathcal{C}$ and an equivalence $Fc \simeq d$.

So, we know what our weak equivalences and fibrations should be. The axioms of a model category imply that IF a model structure with such weak equivalences and fibrations exist then it is unique, however, it obviously does not directly imply that such a model structure even exists. For that we need a strong result about cofibrantly generated model strutures. In its most general form it can be found for example in [Hi09, Theorem 11.3.1].

Here is an idea about the statement of the theorem. The key input is to find a two sets of morphisms I, J which have the following properties:

(1) The maps in I and J satisfy some set-theoretical boundedness condition (in particular on their domains).
(2) A functor is a fibration if and only if it has the right lifting property with respect to maps in J.
(3) A functor is a fibration and weak equivalence if and only if it has the right lifting property with respect to maps in I.
(4) The trivial cofibrations are exactly the maps that have the left lifting property with respect to fibrations.

Then there exists a cofibrantly generated model category with the desired fibrations and weak equivalences.

So, the key insight of the proof is to correctly identify the set of maps I, J. This has been done by Julie Bergner in her work [Be07], which is exactly why the model structure is now commonly known as the Bergner model structure. Interestingly enough, the choices of sets make a lot of sense if we have seen simplicial sets before.

(1) The set I consists of maps
$$\Sigma\partial\Delta^n \to \Sigma\Delta^n$$
$$\{\emptyset\} \to \{x\}$$
(2) The set J consists of maps
$$\Sigma\Lambda^n_i \to \Sigma\Delta^n$$
$$\{x\} \to \Sigma I$$

So, we are "locally" lifting against inner horns for fibrations and boundaries for trivial fibrations.

Using a careful model category approach, we get following theorem:

Theorem 2.5 ([Be07]). *There is a unique model structure on the category of simplicially enriched categories, which satisfies following conditions:*

(1) Fibrations are the functors that are isofibrations and local Kan fibrations.
(2) Weak equivalenes ar the fully faithful essentially surjective functors
(3) It is a proper model structure

2.3 Quasi-Categories. We now want to see that the category $s\mathbb{S}et$ of simplicial sets has a model structure where the fibrant cofibrant objects are exactly quasi-categories. Again, let us think about what the desired data is that we can easily determine. First of all the fibrant objects are quasi-categories, which are simplicial sets which have the right lifting property with respect to inner horns. Also the rule is that whenever we use simplicial sets then the cofibrations are just monomorphisms. However, we haven't determined fibrations and weak equivalences (and, as a matter of fact, they cannot be easily determined).

In order to prove that this gives us a model structure, we use Cisinski's approach to model categories on presheaf toposes [Ci06]. Here is how it goes.

Definition 2.6. Let $f : A \to B$ and $g : C \to D$ be two maps of simplicial sets. We define the pushout product $f\Box g$ as the universal map
$$A \times D \coprod_{A\times C} C \times B \to B \times D$$

For notation sake let
$$\mathcal{B} = \{\partial\Delta^n \to \Delta^n\}$$

Now we make following construction:
$$A_J(S) = \mathcal{B}\Box(0 \hookrightarrow J) \cup S \cup S\Box(\partial I \to I) \cup S\Box(\partial I \to I)\Box(\partial I \to I) \cup ...$$

With those definition we can now describe Cisinski's way of constructing model structures on simplicial sets.

Theorem 2.7. *Let S be a set of monomorphisms in $s\mathbb{S}et$ then there exists a unique left proper model structure on $s\mathbb{S}et$ such that*

- *Cofibrations are monomorphisms*
- *Fibrant objects are characterized by right lifting property with respect to maps in $A_J(S)$.*

We want to apply this theorem. Let S be the set of spine inclusions

$$\Delta^1 \coprod_{\Delta^0} ... \coprod_{\Delta^0} \Delta^1 \to \Delta^n$$

Here is a cool fact: If S is the set of spine inclusions then $A_J(S)$ is the set of inner horns (or an equivalent collection). The idea of this is attributed to Joyal but for a proof see [Ar14, Theorem 5.20].

Thus, we immediately get following result:

Theorem 2.8. *There is a left-proper model structure on simplicial sets with fibrant object quasi-categories and cofibrations monomorphisms.*

The construction we gave here is not the original one. The original construction is due to Joyal and therefore it now known as the Joyal model structure.

Remark 2.9. There is an alternative approach due to Lurie [Lu09, Theorem 2.2.5.1] which relies on his fundamental result [Lu09, Proposition A.2.6.15].

2.4 Complete Segal Spaces. Finally, we want to show there is a model structure on simplicial spaces that helps us study complete Segal spaces. Again, we first observe what the desirable data is. Again, we are studying simplicial objects so the cofibrations will be monomorphisms. Moreover, the fibrant objects are the complete Segal spaces.

However, instead of directly constructing a model structure, we observe that a complete Segal space is a Reedy fibrant simplicial space which satisfies two locality conditions. Thus we will start by constructing a model structure where the fibrant objects are the Reedy fibrant simplicial spaces and then localize to get the desired model structure.

In order to define a Reedy model structure we have to start with a Reedy category I. This gives us following general theorem.

Theorem 2.10. *Let $\mathcal{M}$ be a model category. Then there exists a model structure on the category $\mathcal{M}^{\Delta^{op}}$ such that a map $f : X \to Y$ is a ...*

(1) ... Reedy weak equivalence if for all α in I

$$X_\alpha \to Y_\alpha$$

is a weak equivalence in $\mathcal{M}$.

(2) ... a Reedy cofibration if for all α in I, the relative latching object

$$X_\alpha \coprod_{L_\alpha X} L_\alpha Y \to Y_\alpha$$

is a cofibration in $\mathcal{M}$.

(3) ... a Reedy fibration if for all α in I, the relative matching object

$$X_\alpha \to Y_\alpha \times_{M_\alpha Y} M_\alpha X$$

is a fibration in $\mathcal{M}$.

A proof of this general statement can be found for example in [Hi09, Theorem 15.3.4]. The statement is not very helpful in its most general form as it is often quite difficult to compute these matching objects and latching objects. So, we will rather restrict to the special case we care about.

First, we have following cool computational fact about latching objects.

Lemma 2.11. *Let $\mathcal{M}$ be a model structure such that the cofibrations are monomorphisms. Then the Reedy cofibrations in $\mathcal{M}^I$ are level-wise monomorphisms.*

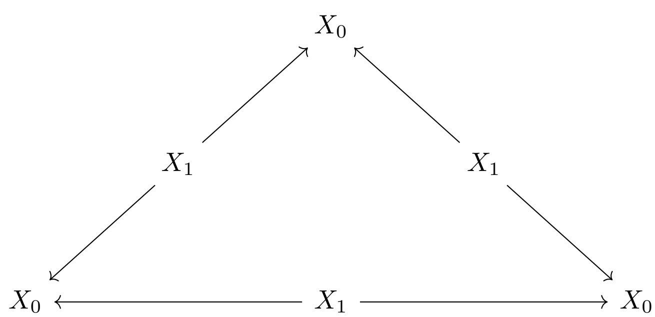
This gives us the desired cofibrations. There is a similar result for matching objects in one special case.

Lemma 2.12. *Let $I = \Delta^{op}$ (which is a Reedy category and in fact the most famous Reedy category). Then, the matching object can be described as the limit diagram given by the boundary of the n-simplex.*

So, in particular

$$M_0 X = X_0 \times X_0$$

and $M_1 X$ is the limit of the diagram



We can now use these results on the one example we really care about:

Theorem 2.13. *There is a model structure on the category of simplicial spaces $s\mathcal{S}$ such that a map $f : X \to Y$ is a ...*

(1) ... Reedy weak equivalence if for all n

$$f_n : X_n \to Y_n$$

is a Kan equivalence.

(2) ... Reedy cofibration if for all n

$$f_n : X_n \to Y_n$$

is a monomorphism.

(3) ... *Reedy fibration if for all* n

$$X_n \to Y_n \times_{Map(\partial F(n),X)} Map(\partial F(n), Y)$$

is a Kan fibration.

Here we used the observation that the matching object can be very explicitly described as

$$M_n X = Map(\partial F(n), X)$$

Thus we get our desired Reedy model structure on simplicial spaces. We can now move on to construct complete Segal spaces. For that we need the theory of Bousfield localizations. Here is one such statement given in [Re01, Proposition 9.1]

Proposition 2.14. *Given am inclusion* $f : A \to B$ *in* $s\mathcal{S}$*, there exists a cofibrantly generated, simplicial model category structure on* $s\mathcal{S}$ *with the following properties:*

(1) *the cofibrations are exactly the inclusions,*
(2) *the fibrant objects (called* f*-local objects) are exactly the Reedy fibration* W *in* $s\mathcal{S}$ *such that*

$$Map_{s\mathcal{S}}(B, W) \to Map_{s\mathcal{S}}(A, W)$$

is a Kan equivalence of spaces,
(3) *the weak equivalences (called* f*-local weak equivalences) are exactly the maps* $g : X \to Y$ *such that for every* f*-local object* W*, the induced map*

$$Map_{s\mathcal{S}}(Y, W) \to Map_{s\mathcal{S}}(X, W)$$

is a Kan equivalence, and
(4) *a weak equivalences (fibrations) between two objects* f*-local objects are exactly the Reedy weak equivalences (Reedy fibrations).*

This is called the Bousfield localization of the Reedy model structure. Note the statement still holds if we apply it to a set of monomorphisms.

We can now immediately apply this Bousfield localization to get a model structure for complete Segal spaces.

Theorem 2.15. *There exists a simplicial, left proper model structure on* $s\mathcal{S}$ *such that*

(1) *the cofibrations are monomorphisms,*
(2) *fibrant objects are the complete Segal spaces,*
(3) *A map* $f : A \to B$ *is a CSS equivalence if and only if*

$$Map(B, W) \to Map(A, W)$$

is a Kan equivalence for every CSS W*.*

Proof. It suffices to apply Bousfield localization theory to the set

$$\{G(n) = F(1) \coprod_{F(0)} ... \coprod_{F(0)} F(1) \hookrightarrow F(n) : n \geq 2\} \cup \{F(0) \hookrightarrow F(1) \coprod_{Z(3)} F(3)\}$$

where

$$Z(3) = F(1) \coprod_{F(0)} F(1) \coprod_{F(0)} F(1)$$

and can be depicted as

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

□

We are now ready to move on and compare these model structures.

From one Model to the Next

3.1 Quillen adjunctions and Quillen equivalences. We have defined what a model structure is, but not yet discussed how to compare them. This happens via Quillen adjunctions.

Definition 3.1. Let $\mathcal{M}, \mathcal{N}$ be two model categories. An adjunction

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

is a Quillen adjunction if one of the following conditions holds:

(1) F preserves cofibrations and trivial cofibrations.
(2) G preserves fibrations and trivial fibrations.

A Quillen adjunction is a Quillen equivalence if the adjunction is an equivalence.

Definition 3.2. A Quillen adjunction

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

is a Quillen equivalence if the derived unit and derived counit maps are equivalences.

One cool fact is that there is a useful lemma for determining Quillen equivalences.

Proposition 3.3 ([JT07, Proposition 7.17])**.** *For a given Quillen adjunction*

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

the following are equivalent

(1) It is a Quillen equivalence.
(2) The map $FLGX \to X$ is a weak equivalence for every fibrant-cofibrant object X in $\mathcal{N}$, where $LGX \to GX$ denotes a cofibrant replacement of GX and the functor F reflects weak equivalences between cofibrant objects.

(3) *The map $Y \to GRFY$ is a weak equivalence for every fibrant-cofibrant object Y in $\mathcal{M}$, where $FY \to RFY$ is the fibrant replacement of FY and the functor G reflects weak equivalences between fibrant objects.*

This last proposition is a very effective concrete theorem with conditions that can be concretely verified.

3.2 Nerves and Necklaces. We want to see how we can compare quasi-categories and Kan enriched categories. This is done via the adjunction $(\mathfrak{C}, N_\Delta)$. How, can we construct such an adjunction? If $\mathfrak{C}$ exists then it needs to commute with colimits because it is a left adjoint. But every simplicial set is a colimit of its simplices, concretely

$$S = \operatorname*{colim}_{\Delta^n \to X} \Delta^n = \operatorname{colim}(\Delta_{/X} \to \Delta \to s\mathcal{S}et)$$

Thus it suffices to determine $\mathfrak{C}[\Delta^n]$.

Naively, we want $\mathfrak{C}[\Delta^n]$ to be a simplicial category which is equivalent to the poset $[n]$. Thus, $\mathfrak{C}[\Delta^n]$ should be a simplicially enriched category with $n+1$ objects $\{0, 1, ..., n\}$ such that

$$Map_{\mathfrak{C}[\Delta^n]}(i,j) = \begin{cases} \emptyset & j < i \\ \text{contractible} & i \leq j \end{cases}$$

The key is to pick the the contractible space $Map_{\mathfrak{C}[\Delta^n]}(i,j)$ such that it is sufficiently free, in the sense that no compositions should not strictly coincide, but rather there should be a path.

Example 3.4. Let us analyze some low-dimensional cases:

(1) $\mathfrak{C}[\Delta^0]$ is a simplicially enriched category with one object i.e. $[0]$.
(2) $\mathfrak{C}[\Delta^1]$ is a simplicially category with one unique non-trivial morphisms i.e. $[1]$.
(3) $\mathfrak{C}[\Delta^2]$ is a simplicially enriched category with 3 objects: $\{0, 1, 2\}$. From the previous case we already know:

$$Map_{\mathfrak{C}[\Delta^2]}(0,1) = \{01\}$$
$$Map_{\mathfrak{C}[\Delta^2]}(1,2) = \{12\}$$

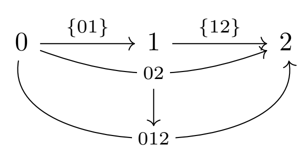
However, we do not have

$$Map_{\mathfrak{C}[\Delta^2]}(0,2) \neq \{02\}$$

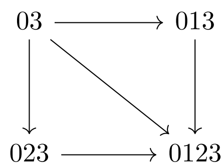
That is because, there is a composition of 01 and 12, which lives in $Map_{\mathfrak{C}[\Delta^2]}(0,2)$ and we denote by 012. We do not want $02 = 012$, rather we want there to be a path. Thus we have

$$Map_{\mathfrak{C}[\Delta^2]}(0,2) = \Delta^1$$

which we can depict by



(4) $\mathfrak{C}[\Delta^3]$ is a simplicially enriched category with 4 objects: $\{0,1,2,3\}$. We only need to understand the case $Map_{\mathfrak{C}[\Delta^3]}(0,3)$. There are four ways to get a map from 0 to 3 via composition. First there is just a map 03, then going through 1, 013, also 023 and finally 0123. with homotopies we can depict this as



The examples we have checked suggest a very clear definition:

$$Map_{\mathfrak{C}[\Delta^n]}(i,j) = N\{S \subseteq [i,j] : i,j \in S\}$$

if $i \leq j$ and otherwise it is empty.

Thus we can now define

$$\mathfrak{C}[S] = \operatorname*{colim}_{\Delta^n \to X} \mathfrak{C}[\Delta^n]$$

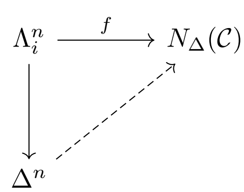
and

$$N_\Delta(\mathcal{C})_n = Hom(\mathfrak{C}[\Delta^n], \mathcal{C})$$

This functor is interesting from a model category perspective.

Proposition 3.5 ([Lu09, Proposition 1.1.5.10])**.** *Let $\mathcal{C}$ be a Kan enriched category. Then $N_\Delta(\mathcal{C})$ is a quasi-category.*

Proof. We need to check the inner horn lifting conditions for $N_\Delta(\mathcal{C})$.



In order show such a lift exist we need a better understanding of $\mathfrak{C}[\Lambda^n_i]$:

- The objects are the same as $\mathfrak{C}[\Delta^n]$.
- If $|i-j| < n$ then
$$Map_{\mathfrak{C}[\Lambda^n_k]}(i,j) = Map_{\mathfrak{C}[\Delta^n]}(i,j)$$
but
$$Map_{\mathfrak{C}[\Lambda^n_k]}(0,n) \cong [\partial((\Delta^1)^{n-2}) \times \Delta^1] \coprod_{\partial((\Delta^1)^{n-2})} (\Delta^1)^{n-2}$$
i.e. it is a sub-cube with the interior and one face removed.

Thus the existence of a lift reduces to the existence of a lift in the diagram

$$\begin{array}{ccc} \partial((\Delta^1)^{n-2}) \times \Delta^1 \coprod_{\partial((\Delta^1)^{n-2})} (\Delta^1)^{n-2} \cong Map_{\mathfrak{C}[\Lambda^n_k]}(0,n) & \xrightarrow{f(0,n)} & Map_{\mathcal{C}}(f(0),f(n)) \\ \big\downarrow \simeq & \nearrow & \\ (\Delta^1)^{n-1} = Map_{\mathfrak{C}[\Delta^n]}(0,n) & & \end{array}$$

By assumption $Map_{\mathcal{C}}(f(0), f(n))$ is a Kan complex and so we only need the left hand map to be a trivial cofibration, which clearly holds (it is an inclusion and domain and codomain are contractible). □

This proof in particular illustrates that studying the functor $\mathcal{N}_\Delta$ is at least possible. This same is not true for $\mathfrak{C}[]$, because it is defined as a colimit in the category of simplicial categories. Those are notoriously difficult and thus have been studied separately by many other authors [Ri11].

The next goal is to give one such description via Duggar and Spivak using the theory of necklaces [DS11].

Definition 3.6. For two simplices Δ^n and Δ^m, we define $\Delta^n \vee \Delta^m$ as the pushout:

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{\{n\}} & \Delta^n \\ \{0\}\big\downarrow & & \big\downarrow \\ \Delta^m & \longrightarrow & \Delta^n \vee \Delta^m \end{array}$$

Notation 3.7. A **necklace** is a simplicial set T of the form

$$T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$$

Example 3.8. A necklace generalizes several important classes of simplicial sets:

(1) If $k = 1$, then we just get a simplex.
(2) If $n_i = 1$ for all i, then the necklace is just the spine.

Necklaces form an important category.

Definition 3.9. Let $\mathcal{N}ec$ be the category with objects necklaces and morphisms, morphisms of simplicial sets that preserve the initial and terminal points.

Notice there is an obvious inclusion functor $\mathcal{N}ec \hookrightarrow s\mathcal{S}et_{*\coprod */}$, the category of simplicial sets, with two distinguished points.

We now want to use the category $\mathcal{N}ec$ to construct a category enriched over categories. Let S be a simplicial set, for any two 0-simplices a, b define $\mathcal{N}ec_{/S_{a,b}}$ as the pullback of categories

$$\begin{array}{ccc} \mathcal{N}ec_{/S_{a,b}} & \longrightarrow & (s\mathcal{S}et_{*\coprod */})_{/S_{a,b}} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{N}ec & \longrightarrow & s\mathcal{S}et_{*\coprod */} \end{array}$$

An object in the category $\mathcal{N}ec_{/S_{a,b}}$ is a necklace T and a map of simplicial sets $T \to S$ that takes the initial point to a and the final point to b. This construction comes with an evident composition map: For three points a, b, c there is a functor:

$$\mathcal{N}ec_{/S_{a,b}} \times \mathcal{N}ec_{/S_{b,c}} \to \mathcal{N}ec_{/S_{a,c}}$$

which takes $(T_1, T_1 \to S)$, $(T_2, T_2 \to S)$ to $(T_1 \vee T_2, T_1 \vee T_2 \to S)$, using the fact that $T_1 \vee T_2$ is again a necklace. Thus, this gives us a category enriched over categories, where the objects are the 0-cells of S and the morphisms categories are the categories $\mathcal{N}ec_{/S_{a,b}}$.

How do we get a simplicial category out of this? We simply apply the nerve.

Definition 3.10. Let the simplicially enriched category $\mathfrak{C}^{nec}[S]$ defined by having objects S_0 and morphism simplicial set:

$$Map_{\mathfrak{C}^{nec}}(a, b) = N(\mathcal{N}ec_{/S_{a,b}})$$

The key result in the paper [DS11] is that this definition is equivalent to the colimit formula given above.

Theorem 3.11. *There is a natural zig-zag of weak equivalences of simplicial categories between $\mathfrak{C}^{nec}[S]$ and $\mathfrak{C}[S]$, for all simplicial sets S.*

We end this subsection by noting that $(\mathfrak{C}, N_\Delta)$ is indeed a Quillen equivalence, which we will not prove.

Theorem 3.12 ([Lu09, Theorem 2.2.5.1])**.** *The adjunction*

$$s\mathcal{S}et \underset{N_\Delta}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathcal{C}at_\Delta$$

is a Quillen equivalence between the Joyal model structure and the Bergner model structure.

3.3 Segal Spaces vs. Quasi-Categories. This part is completely dedicated to the work by Joyal and Tierney [JT07] who carefully study the comparison between complete Segal spaces and quasi-categories (and Segal categories, which we will not mention).

Concretely, we want to show that there is a chain of Quillen equivalences:

$$s\mathcal{S} \underset{t^!}{\overset{t_!}{\rightleftarrows}} s\mathcal{S}et \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} s\mathcal{S}$$

This equivalence is very helpful in understanding what $(\infty, 1)$-category theory is about and why quasi-categories are so powerful. Any argument given here can be traced back to this paper, but some of the notation, order or arguments have been adjusted.

First some background stuff that will be important later on:

Remark 3.13. Previously we defined the completeness condition for a Segal space to be the fact that the following is a pullback square

$$\begin{array}{ccc} W_0 & \longrightarrow & W_3 \\ \big\downarrow & \lrcorner & \big\downarrow \\ W_1 & \longrightarrow & W_1 \times_{W_0} W_1 \times_{W_0} W_1 \end{array}$$

Here we will use the fact that this is equivalent to the map of spaces:

$$Map(E(1), W) \to Map(F(0), W) = W_0$$

being an equivalence, where $E(1)$ the discrete simplicial space defined, which is given by horizontally embedding the simplicial set $I[1]$ and plays the role of the "free isomorphism" of complete Segal spaces. More formally, $E(1) = p_1^*(I[1])$, where p_1^* is defined below.

A more careful analysis of completeness follows in the next section.

Next we will introduce the functors I and J. The inclusion functor $inc : \mathcal{K}\mathrm{an} \to \mathcal{QC}\mathrm{at}$ has both a left and right adjoint: the left adjoint is the "groupoidification" and the right adjoint the "core". Concretely, we have diagrams

$$\mathcal{QC}\mathrm{at} \underset{inc}{\overset{I}{\rightleftarrows}} \mathcal{K}\mathrm{an} \underset{J}{\overset{inc}{\rightleftarrows}} \mathcal{QC}\mathrm{at}$$

We can compose the adjunctions to the get an adjunction of quasi-categories (where we ignore the inclusions)

$$\mathcal{QC}\mathrm{at} \underset{J}{\overset{I}{\rightleftarrows}} \mathcal{QC}\mathrm{at}$$

Let us give some concrete information about this adjunction.

First of all, (I, inc) is a Quilled adjunction between the Kan model structure and the Joyal model structure, but (inc, J) is not. So (I, J) is also not a Quillen adjunction. That is why we specifically wrote it for the categories and not the corresponding model categories. The problem is, although inc preserves equivalences between fibrant objects, it does not preserve equivalences between cofibrant objects.

Recall we defined $I[n]$ to be the nerve of the groupoid with exactly one morphism between any two objects (seemingly also called a lense). The functor J takes a quasi-category to its underlying core. Concretely $J(S)_n = Hom(I[n], S)$.

We have following facts about J:

(1) The map J takes a categorical fibration between quasi-categories to a Kan fibration between Kan complexes.
(2) The map J takes a categorical equivalence of quasi-categories to a Kan equivalence of Kan complexes.

These facts are given in [JT07, Proposition 1.16].

Notation 3.14. Throughout we will often simplify $I[\Delta^n]$ to $I[n]$.

First we want to define these relevant functors. The adjunction (p_1^*, i_1^*) is defined quite easily. For a given simplicial set S, we define the simplicial space $p_1^*(S)$ as

$$p_1^*(S)_{nl} = S_n$$

and for a given simplicial space X, we define $i_1^*(X)$ as

$$i_1^*(X)_n = X_{n0}$$

More rigorously, the adjunction is characterized uniquely by the fact that

$$p_1^*(\Delta^n) = F(n)$$

and the fact that every simplicial set is the colimit of the simplices. So, basically i_1^* remembers the first row of the simplicial space.

Now we describe the adjunction $(t_!, t^!)$. Again, we use the fact that a functor out of simplicial spaces is determined by the objects $F(n)$ and Δ^l. We define

$$t_!(F(n)) = \Delta^n$$

$$t_!(\Delta^l) = I[n]$$

where recall that $I[n]$ is the nerve of the groupoid with n objects and a unique morphism between any two objects. This also tells us how to define $t^!$, namely

$$t^!(S)_{nl} = Hom(\Delta^n \times I[l], S)$$

The way to think about this functor is that it takes a simplicial set S to the simplicial space, which row-wise has the same data, but column-wise only has the data of the underlying Kan complex.

The goal is to show that these adjunctions give us Quillen equivalences. Here are the steps:

(1) Show that both are Quillen adjunctions.
(2) Show that the adjunction (p_1^*, i_1^*) is a Quillen equivalence.
(3) Show that the composition is also a Quillen equivalence and deduce the result by 2 out of 3.

In order to prove we have Quillen adjunctions we will use following lemma.

Lemma 3.15 ([JT07, Proposition 7.15])**.** *(F, G) is a Quillen adjunction if and only if*

(1) F takes cofibrations to cofibrations.
(2) G takes fibrations between fibrant objects to fibrations

Lemma 3.16 ([JT07, Proposition 4.7]). *(p_1^*, i_1^*) is a Quillen adjunction.*

Proof. We will use the lemma above. First it is clear that p_1^* preserves cofibrations as they are all just inclusions. Thus we only need to prove that for every CSS fibration between two CSS $f : V \to W$, $i_1^*(f)$ is a fibration in the Joyal model structure.

General fibrations in the Joyal model structures can be tricky, however, there is one special case that is easier to understand: A fibration between quasi-categories is an inner fibration with lift with respect to the free equivalence. This means it lifts with respect to the maps:

$$\Lambda^n_i \to \Delta^n; 0 < i < n$$
$$\Delta^0 \to I[1]$$

Thus the first step of the proof is to show that $i_1^*(W)$ is a quasi-category. For that we need to prove that the following diagram lifts

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & i_1^*(W) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

using an adjunction this is equivalent to having a lift in the diagram

$$\begin{array}{ccc} p_1^*(\Lambda^n_i) & \longrightarrow & W \\ \downarrow & \nearrow & \\ p_1^*(\Delta^n) & & \end{array}$$

By definition $p_1^*(\Delta^n) = F(n)$. Thus we only need to observe that $p_1^*(\Lambda^n_i)$ is a colimit of $F(n)$ and $G(n)$. We will show this for $n = 2, 3$ and leave the general case as an exercise. First we have $p_1^*(\Lambda^2_1) = G(2)$. Next we have the following two pushout squares

$$\begin{array}{ccc} G(2) \coprod G(2) & \longrightarrow & F(2) \coprod F(2) \\ {\scriptstyle 012 \coprod 123} \downarrow & & \downarrow \\ G(3) & \longrightarrow & T \end{array} \qquad \begin{array}{ccc} G(2) & \longrightarrow & F(2) \\ {\scriptstyle 013} \downarrow & & \downarrow \\ T & \longrightarrow & p_1^*(\Lambda^3_1) \end{array}$$

The argument for $p_1^*(\Lambda^3_2)$ is similar.

Thus we only need to show that $p_1^*(f) : p_1^*(V) \to p_1^*(W)$ is an isofibration. This means we need to show that it lifts against maps $\Lambda^n_i \to \Delta^n$ and $\Delta^0 \to I$. But this is equivalent to $f : V \to W$ having lifts against $i_1^*(\Lambda_i^n) \to i_1^*(\Delta^n)$ and $i_1^*(\Delta^0) \to i_1^*(I)$. We already observed that $i_1^*(\Lambda_i^n) \to i_1^*(\Delta^n)$ is an equivalence, as it can be constructed as a pushout of maps $G(n) \to F(n)$. Finally, $i_1^*(\Delta^0) = F(0)$ and $i_1^*(I) = E(1)$ is the analogue of the walking isomorphism and so liftings exist because of the completeness condition. □

Remark 3.17 ([JT07, Proposition 4.4]). The lemma above has following direct generalization: Let $i_n^* : s\mathcal{S} \to s\mathcal{S}et$ be the functor that restricts a simplicial space to its n-th row. Then i_n^* takes a complete Segal space to a quasi-category. It suffices to observe that the left adjoint is defined as $p_n^*(\Delta^l) = F(l) \times \Delta^n$ and then use the argument given above. In fact the functors $i_1^*, i_2^*, ...$ have an even stronger relationship, which we will sketch here:

If W is a complete Segal space, then any map $i_{n+1}^*(W) \to i_{m+1}^*(W)$ induced by a map $[m] \to [n]$ is an equivalence of quasi-categories. It suffices to check this for the case $i_1^*(W) \to i_{n+1}^*(W)$ induced by the unique map $[n] \to [0]$. This is a consequence of the fact that W is complete. Notice the simplicial set $i_{n+1}^*(W)$ has the form

$$W_{0n} \leftrightarrows W_{1n} \leftrightarrows W_{2n} \leftrightarrows \cdots$$

The completeness condition tells us that elements in W_{0n} corresponds to n composable equivalences i.e. a diagram

$$I[n] \to i_1^*(W)$$

Simiarly, W_{1n} corresponds to a diagram

$$\Delta^1 \times I[n] \to i_1^*(W)$$

Thus we can think of $i_{n+1}^*(W) \simeq i_1^*(W)^{I[n]}$ and the map above as the unique inclusion map

$$i_1^*(W) \to i_1^*(W)^{I[n]}$$

which is clearly an equivalence as $I[n]$ is a contractible quasi-category.

The previous Remark has an even stronger implication, which we will not prove:

Lemma 3.18 ([JT07, Theorem 4.5]). *A map of simplicial spaces $X \to Y$ is a CSS equivalence if and only if the maps*

$$i_{n+1}^*(X) \to i_{n+1}^*(Y)$$

is a categorical equivalence of quasi-categories for every $n \geq 0$.

The precise argument needs a comparison between the "horizontal" and "vertical" Reedy model structure, which we shall not do.

Lemma 3.19 ([JT07, Theorem 2.12, Theorem 3.3, Theorem 4.12]). *The adjunction $(t_!, t^!)$ is a Quillen adjunction.*

Proof. Recall $t_!$ is horizontally the identity and vertically a Kan fibrant replacement, both of which preserve inclusions (which are our cofibrations). Thus we need to show that $t^!$ takes an isofibration $f : S \to T$ between between quasi-categories to a CSS fibration $t^!(f) : t^!(S) \to t^!(T)$.

First we show that $t^!(f)$ is a Reedy fibration. For that we need to prove that the diagram lifts

$$\begin{array}{ccc} \partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l} F(n) \times \Lambda^l & \longrightarrow & t^!(S) \\ \downarrow & \nearrow & \downarrow \\ F(n) \times \Delta^l & \longrightarrow & t^!(T) \end{array}$$

by adjunction this is equivalent to

$$\begin{array}{ccc} t_!(\partial F(n) \times \Delta^l \coprod_{\partial F(n) \times \Lambda^l} F(n) \times \Lambda^l) & \longrightarrow & S \\ \downarrow & \nearrow & \downarrow \\ t_!(F(n) \times \Delta^l) & \longrightarrow & T \end{array}$$

Direct computation gives us the diagram

$$\begin{array}{ccc} \partial \Delta^n \times I[\Delta^l] \coprod_{\partial \Delta^n \times I[\Lambda^l]} \Delta^n \times I[\Lambda^l]) & \longrightarrow & S \\ \downarrow & \nearrow & \downarrow \\ \Delta^n \times I[\Delta^l] & \longrightarrow & T \end{array}$$

which means we need to observe that

$$\partial \Delta^n \times I[\Delta^l] \coprod_{\partial \Delta^n \times I[\Lambda^l]} \Delta^n \times I[\Lambda^l]) \to \Delta^n \times I[\Delta^l]$$

is a trivial cofibrations. This follows from the general observation that if we have two cofibrations:

$$f : A \to B$$

$$g : C \to D$$

such that one is a trivial cofibration, then the map

$$f \square g : A \times B \coprod_{A \times C} B \times C \to B \times D$$

is a trivial cofibration.

Notice we really used the $I[]$ here as Λ^l is an arbitrary horn and not just an inner horn.

Next we prove that $t^!(S)$ is a complete Segal space, where S is a quasi-category. We have already shown that it is Reedy fibrant, thus we only have to check the Segal condition and completeness condition. We need to prove that

$$Map(F(n), t^!(S)) \to Map(G(n), t^!(S))$$

is a trivial Kan fibration. We have

$$Map(F(n), t^!(S))_k = Map(F(n) \times \Delta^k, t^!(S)) \cong Map(t_!(F(n) \times \Delta^k), S) = Map(\Delta^n \times I[\Delta^k], S)$$

Similarly, we have

$$Map(G(n), t^!(S))_k = Map(G(n) \times \Delta^k, t^!(S)) \cong Map(t_!(G(n) \times \Delta^k), S) = Map(Sp^n \times I[\Delta^k], S)$$

Thus, we need to show that the map

$$Map(I[\Delta^k], S^{Sp^n}) \to Map(I[\Delta^k], S^{\Delta^n})$$

is an equivalence, which by adjunction means

$$Map(\Delta^k, J(S^{Sp^n})) \to Map(\Delta^k, J(S^{\Delta^n}))$$

which means, we need to show the map of cores

$$J(S^{\Delta^n}) \to J(S^{Sp_n})$$

is a Kan equivalence. This follows from the fact that the map

$$S^{\Delta^n} \to S^{Sp_n}$$

is a categorical equivalence of quasi-categories.

Finally, we need to prove that $t^!(S)$ is complete. We need to show

$$Map(E(1), t^!(S)) \to Map(F(0), t^!(S))$$

is a trivial Kan fibration. We already showed that

$$Map(F(0), t^!(S))_k = Map(I[\Delta^k], S)$$

and

$$Map(E(1), t^!(S))_k = Map(E(1) \times \Delta^k, t^!(S)) = Map(t_!(E(1) \times \Delta^k), S) = Map(I[\Delta^1] \times I[\Delta^k], S) \cong Map(I[\Delta^k], S^{I[\Delta^1]})$$

Thus we need to prove that

$$J(S) \to J(S^{I[\Delta^1]})$$

is a Kan equivalence, which again follows from the fact that

$$S \to S^{I[\Delta^1]}$$

is a categorical equivalence.

Finally, we prove that $t^!(f)$ is a CSS fibration. However a CSS fibration between CSS is just a Reedy fibration (one of the many benefits of being a Bousfield localization). Thus it suffices to prove we $t^!(f)$ is a Reedy fibration, which we have already shown. Hence we are done. □

We are now in a position to prove these are Quillen equivalences. We will use following nice way of characterizing a Quillen equivalence:

Proposition 3.20 ([JT07, Proposition 7.17, Proposition 7.22]). *A Quillen adjunction*

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

is a Quillen equivalence if one of the following conditions holds:

(1) *The following two conditions hold:*
   (a) *For every fibrant-cofibrant object $Y \in \mathcal{N}$ the derived counit map*
   $$FLGY \to FGY \to Y$$
   is an equivalence in $\mathcal{N}$.
   (b) *For every fibrant-cofibrant object $X \in \mathcal{M}$ the derived unit map*
   $$X \to GFX \to GRFX$$
   is an equivalence in $\mathcal{M}$.
(2) *The following two conditions hold:*
   (a) *For every fibrant-cofibrant object $Y \in \mathcal{N}$ the derived counit map*
   $$FLGY \to FGY \to Y$$
   is an equivalence in $\mathcal{N}$.
   (b) *F reflects equivalences between cofibrant objects*
(3) *The following two conditions hold:*
   (a) *For every fibrant-cofibrant object $X \in \mathcal{M}$ the derived unit map*
   $$X \to GFX \to GRFX$$
   is an equivalence in $\mathcal{M}$.
   (b) *G reflects equivalences between fibrant objects.*

First we show that (p_1^*, i_1^*) is a Quillen equivalence [JT07, Theorem 4.11]. Very concretely, we need to prove the following facts:

(1) The derived counit map
$$p_1^* i_1^* W \to W$$
is a CSS equivalence for every complete Segal space W (the cofibrant replacement is not necessary as everything is cofibrant).
(2) The derived unit map
$$S \to i_1^* p_1^* S \to i_1^* R p_1^* S$$
is a categorical equivalence for every quasi-category S.

Let us start with the first one. This is the content of [JT07, Proposition 4.7]. By Remark 3.18 it suffices to check that we have a level-wise equivalence of quasi-categories
$$W_{\bullet 0} \to W_{\bullet n}$$
But this was shown in Remark 3.17.

Now we prove the second argument. First observe that $i_1^* p_1^* S = S$. The main problem is the fibrant replacement. Clearly $p_1^* S$ is (usually) not a complete Segal space. Actually, why not?

(1) It is Reedy fibrant (every discrete simplicial space is Reedy fibrant).

(2) It does not satisfy the Segal condition (unless it is the nerve of a category).
(3) It does not satisfy the completeness condition (unless it has no non-trivial automorphisms).

Thus the point is to find a fibrant replacement of p_1^*S. Obviously there is an abstract fibrant replacement, given by the axioms of a model category, but we want to do better than that. We want to find a fibrant replacement we can actually control and understand well. This is exactly what Joyal and Tierney do in [JT07, Proposition 4.10]. For a quasi-category S define the simplicial space

$$\Gamma(S)_n = J(S^{\Delta^n})$$

the maximal subgroupoid inside S^{Δ^n}. Thus our simplicial space takes the form:

$$J(S) \leftleftarrows \rightarrow J(S^{\Delta^1}) \cdots J(S^{\Delta^2}) \cdots$$

This object is super fascinating. First of all notice following fact about Γ: For a given simplicial set we have $J(S)_0 = S_0$. Thus $i_1^*(\Gamma(S)) = S$ and moreover, there is a natural inclusion map $p_1^*(S) \to \Gamma(S)_{nl}$.

Remark 3.21. In hindsight I noticed that we should have

$$\Gamma(S)_{nl} = Hom(\Delta^l, J(S^{\Delta^n})) \cong Hom([I[\Delta^l], S^{\Delta^n}) \cong Hom(\Delta^n \times I[\Delta^l], S) \cong t^!(S)$$

This is never explained or mentioned in the paper, so there are some doubts, but it seems correct!

Proposition 3.22 ([JT07, Proposition 4.10]). *Let S be a quasi-category. Then*

(1) $\Gamma(S)$ is a complete Segal space.
(2) The map $p_1^*(S) \to \Gamma(S)$ is a CSS equivalence.

Proof. First we prove that $\Gamma(S)$ is a complete Segal space. We need to prove that the map

$$Map(F(n), \Gamma(S)) \to Map(\partial F(n), \Gamma(S))$$

is a Kan fibration. This means we have to show the map

$$J(S^{\Delta^n}) \to J(S^{\partial\Delta^n})$$

is a Kan fibration. The result now follows from the fact that

$$S^{\Delta^n} \to S^{\partial\Delta^n}$$

is a categorical fibration along with fact 2 above (we used the fact $\partial\Delta^n \to \Delta^n$ is a cofibration).

Next we want to show $\Gamma(S)$ is a Segal space. For that we need to prove that

$$Map(F(n), \Gamma(S)) \to Map(G(n), \Gamma(S))$$

is a trivial Kan fibration. Again this simplifies to

$$J(S^{\Delta^n}) \to J(S^{Sp^n})$$

Again, the result follows from fact (2) and (3) above and that

$$S^{\Delta^n} \to S^{Sp^n}$$

is a trivial categorical fibration (we used the fact that $Sp^n \to \Delta^n$ is a trivial cofibration).

Finally we need to show $\Gamma(S)$ is complete. That means proving that

$$Map(E(1), \Gamma(S)) \to Map(F(0), \Gamma(S))$$

is a trivial fibration. Again unwinding definitions this means the map

$$J(S^{I[1]}) \to J(S)$$

is a trivial Kan fibration. Again, this follows from the fact that

$$S^{I[1]} \to S$$

is a trivial categorical fibration.

Next we need to show that the map $p_1^*(S) \to \Gamma(S)$ is a CSS equivalence. According to Remark 3.18 it suffices to show the map is a level-wise categorical equivalence of quasi-categories. However, this follows immediately from the fact that the map at level n is given by

$$S \to S^{I[n]}$$

which is obviously an equivalence. □

Thus $\Gamma(S)$ is a fibrant replacement of $p_1^*(S)$. Thus the derived unit map

$$S \to i_1^* p_1^* \to i_1^* R p_1^*(S) = i_1^* \Gamma(S) = S$$

is just the identity, which is clearly an equivalence.

Thus we are done!

We want to move on to the next step. We want to show that the composition adjunction is a Quillen equivalence of the Joyal model structure.

$$\text{s}\mathcal{S}\text{et} \underset{i_1^* t^!}{\overset{t_! p_1^*}{\rightleftarrows}} \text{s}\mathcal{S}\text{et}$$

In fact, we can much better! We have

$$t_! p_1^*(\Delta^n) = t_!(F(n)) = \Delta^n$$

This means the left adjoint is the actual identity functor, which means the right adjoint is also the identity functor. Thus it is obviously a Quillen equivalence.

By 2-out-of-3 then we have that

$$s\mathcal{S} \underset{t^!}{\overset{t_!}{\rightleftarrows}} \text{s}\mathcal{S}\text{et}$$

is a Quillen equivalence.

So, why did we do all of this?

(1) It's nice to see actual (mostly) written out proofs about ∞-categories, where things sometime seem very vague.
(2) It's a nice proof that can be explained in a reasonably elementary language.

(3) It helps clarify the relation between quasi-categories and complete Segal spaces.

Let's be more precise about this last point. What does this result tell us about quasi-categories and complete Segal spaces?

(1) From the perspective of a quasi-category theory, complete Segal spaces is just the data of the various cores of the quasi-category. Indeed, let W be an arbitrary complete Segal space. Concretely we have:
$$W \simeq \Gamma(i_1^*(W))$$
(2) From the perspective of complete Segal spaces, a quasi-category is just the first row of a complete Segal space. Concretely for any quasi-category S we have
$$S = i_1^*(\Gamma(S))$$
(3) Thus a complete Segal does not have any extra data. The additional simplicial axis in complete Segal spaces is just keeping track of the various cores $J(S^{\Delta^n})$, which is uniquely determined by the quasi-category S.
(4) On the other hand, the core is an invariant of an $(\infty, 1)$-category and so a complete Segal space is thus an invariant notion, which means we can define it in various contexts. This is clearly not true for quasi-categories (an equivalence of quasi-categories does not imply an isomorphism of simplicial sets).
For example it has been used by Rezk to define a model of (∞, n)-categories via Θ_n-spaces [Re10]. It was based on a working definition of (∞, n)-categories via the complete Segal spaces that an analogous definition was given for Θ_n-sets [Ar14].
More generally, we can use the broad definition to define internal $(\infty, 1)$-category objects (so called complete Segal objects). In particular, it has been used to study higher category theory in homotopy type theory (which leads to so called Rezk types [RS18]).
(5) There is a certain elegance to the adjunctions $(t_!, t^!)$ and (p_1^*, i_1^*). They literally compose to the identity! This is actually not a coincidence. There is a fancy theorem by Toën [To05], which proves that the complete Segal space model structure has one non-trivial automorphisms, namely the one that takes a complete Segal space W to W^{op}. This must then also hold for any Quillen equivalent model structure. Thus the composition $p_1^* t_!$ had to be the identity or the *op* map. Given that we did not change any directions, this implies that the composition shold really be the identity. This is obviously not a proof, but a justification why it is not surprising that the composition is literally the identity. The theory of $(\infty, 1)$-categories is actually quite rigid.
(6) Small sidenote: There is a similar result due to Barwick and Schommer-Pries for (∞, n)-categories, where they prove that there are $(\mathbb{Z}/2)^n$ possible automorphisms [BS11].
(7) This implies in particular that if we we take any other model of $(\infty, 1)$-categories (Segal category, Kan enriched category, ...) and an equivalence to (or from) quasi-categories, then we can get the (essentially unique) equivalence to complete Segal spaces by composing with (p_1^*, i_1^*) or $(t_!, t^!)$.
(8) This also justifies having two different Quillen equivalences. If we think of these objects at the level of $(\infty, 1)$-categories, then we have adjunctions
$$\mathfrak{QCat} \underset{i_1^*}{\overset{Rp_1^*}{\rightleftarrows}} \mathfrak{CSS} \underset{t^!}{\overset{Rt_!}{\rightleftarrows}} \mathfrak{QCat}$$

Notice we need to derive the left adjoints as they do not themselves land in the subcategory of fibrant objects. Now, the fact that these are Quillen equivalences implies that these adjunctions are equivalences. In particular we must have

$$Rp_1^* \simeq t^!$$

$$i_1^* \simeq Rt_!$$

which are not difficult to show if we just did the computations. So, one adjunction is the inverse of the other and so in particular the left adjoints are also a right adjoints and vice versa.

However, notice the opposite is generally not true. If we have been given a Quillen equivalence between two model structures and an inverse adjunction only at the level of underlying $(\infty, 1)$-categories, then that inverse might not come from any Quillen equivalence as it might not preserve fibrations. So, having two concrete Quillen equivalences in both directions goes beyond its theoretical existence and can be used in proofs.

For an interesting example we can see [RV18, E.2.2], where Riehl and Verity use the specific fact that p_1^* preserves finite products to prove that the category $\mathcal{CSS}$ is an ∞-cosmos.

The "Complete" in Complete Segal Spaces

We have now learned enough about $(\infty, 1)$-categories and in particular, quasi-categories and complete Segal spaces to take a closer look at completeness. Here is the general question: Let $\mathcal{C}\text{at}_\infty$ be any model of ∞-category that comes to your mind. Then the following things should be true (probably more than that, but we just need these facts):

(1) The inclusion map $\mathcal{S} \to \mathcal{C}\text{at}_\infty$ has a right adjoint

$$J(-) : \mathcal{C}\text{at}_\infty \to \mathcal{S}$$

(2) There is a cosimplicial object of n-simplices in $\mathcal{C}\text{at}_\infty$:

$$[0] \rightrightarrows [1] \rightrightarrows [2] \rightrightarrows \cdots$$

(3) We have composition, which can be expressed as the fact that

$$\mathcal{C}^{[n]} \to \mathcal{C}^{[1]} \times_{\mathcal{C}} ... \times_{\mathcal{C}} \mathcal{C}^{[1]}$$

is an equivalence of ∞-categories.

(4) There is a notion of objects

(5) There is a notion of mapping space

$$map_{\mathcal{C}}(x, y)$$

(6) A map of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is

I Fully Faithful:

$$map_{\mathcal{C}}(x, y) \to map_{\mathcal{D}}(Fx, Fy)$$

is an equivalence

II Essentially surjective: Every object in $\mathcal{D}$ is equivalent to an object of the form Fc.

We will also call such maps Dwyer-Kan equivalences.

From the first two facts we can deduce the existence of a functor

$$\Gamma : \mathcal{C}\mathrm{at}_\infty \to s\mathcal{S}$$

that takes an ∞-category $\mathcal{C}$ to the simplicial space

$$J(\mathcal{C}) \leftleftarrows J(\mathcal{C}^{[1]}) \leftleftarrows J(\mathcal{C}^{[2]}) \leftleftarrows \cdots$$

Let us assume we know this functor is fully faithful. Now, the question is: What is the essential image of this functor? Clearly it's not everything, so what is it?

We can very immediately make two observations:

(1) The definition given is invariant under level-wise Kan equivalences and so we can always assume that the simplicial space is Reedy fibrant.
(2) The third condition above means we have an equivalence of ∞-categories:
$$\mathcal{C}^{[n]} \to \mathcal{C}^{[1]} \times_{\mathcal{C}} \dots \times_{\mathcal{C}} \mathcal{C}^{[1]}$$
which gives us an equivalence of spaces
$$J(\mathcal{C}^{[n]}) \to J(\mathcal{C}^{[1]}) \times_{J(\mathcal{C})} \dots \times_{J(\mathcal{C})} J(\mathcal{C}^{[1]})$$
But this just means that $\Gamma(\mathcal{C})$ satisfies the Segal condition.

The question is now, is that enough? In order to answer that it is helpful to actually understand the category theory of Segal spaces. What does it mean to be an object or morphism in a Segal space? The main ideas and proofs are due to Rezk [Re01], but the presentation has some differences.

So, let W be a Segal space i.e. a Reedy fibrant simplicial space such that

$$W_n \to W_1 \times_{W_0} \dots \times_{W_0} W_1$$

is an equivalence. We want to now see how we can use the Segal condition to define basic categorical notions.

An object is an element in the set W_{00}. For two objects x, y we define the mapping space $map_W(x, y)$ as the pullback:

$$\begin{array}{ccc} map(x,y) & \longrightarrow & W_1 \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow[(x,y)]{} & W_0 \times W_0 \end{array}$$

Remark 4.1. Cool fact: The fact that W is a Reedy fibrant in particular means that $(s, t) : W_1 \to W_0 \times W_0$ is a Kan fibration, which means that $map(x, y)$ is Kan fibrant (this is quite difficult to prove in the quasi-category model [Re17, Corollary 33.4]).

So, in particular, the objects of $\mathcal{C}$ correspond to $(\Gamma\mathcal{C})_{00}$ and the mapping spaces are given by

$$map_{\mathcal{C}}(x, y) \simeq map_{\Gamma\mathcal{C}}(x, y)$$

defined as the pullback above.

We now have to check whether a Segal space satisfies the last condition stated above, namely whether an equivalence is just an essentially surjective fully faithful functor. Here we can make following important observation about the image of Γ:

Lemma 4.2. *A map $\Gamma F : \Gamma \mathcal{C} \to \Gamma \mathcal{D}$ is an equivalence if one of the following hold:*

(1) There is an inverse $G : \mathcal{D} \to \mathcal{C}$
(2) It is fully faithful and essentially surjective.
(3) The map ΓF is a level-wise Kan equivalence.

Proof. We already know the first two are equivalent (by assumption). So, we will prove the equivalence of these two with the third one. If F is invertible then ΓF also has an inverse, which means we have level-wise inverses, which implies ΓF is level-wise a Kan equivalence.

Now, let us assume ΓF is a level-wise Kan equivalence. Then looking at the diagram

$$\begin{array}{ccc} map_{\Gamma\mathcal{C}}(x,y) & \longrightarrow & map_{\Gamma\mathcal{D}}(Fx,Fy) \\ \simeq\downarrow & & \downarrow\simeq \\ * \times_{\Gamma\mathcal{C}_0 \times \Gamma\mathcal{C}_0} \Gamma\mathcal{C}_1 & \xrightarrow{\simeq} & * \times_{\Gamma\mathcal{D}_0 \times \Gamma\mathcal{D}_0} \Gamma\mathcal{D}_1 \end{array}$$

immediately implies that F is fully faithful (here the vertical maps are equivalences by definition and the bottom horizontal map is an equivalence by assumption). The fact that $J(\mathcal{C}) \to J(\mathcal{D})$ is an equivalence immediately implies that it is also essentially surjective. □

This means the image of Γ consists of all Segal spaces that have the additional property that level-wise equivalence is the same as being fully faithful and essentially surjective. What are those?

Here it is helpful to think of Segal spaces that fail this condition. For example let $F(0) \to W$ be a map of Segal spaces that is fully faithful and essentially surjective, then it should be a level-wise equivalence, which would just mean W is level-wise contractible. In this case, being fully faithful means that there is an object in $x \in W$ with $map_W(x,x) = \{id_x\}$ and essentially surjective means that all objects are equivalent. Is there such a Segal space? If such a Segal space exists then we would need to identify it with $F(0)$ i.e. we need to invert this map.

The answer is clearly yes, namely any groupoid with a unique morphism between any two objects. The smallest non-trivial example is $E(1)$. Recall that we think of it as a discrete simplicial space, which at level n is given by

$$E(1)_n = \{x,y\}^{[n]}$$

each level has exactly two non-degenerate cells giving us the two inverses at each level. They correspond to the two words $(xy)^{n/2}$ and $(yx)^{n/2}$ if n is even and $(xy)^{(n-1)/2}x$ and $(yx)^{(n-1)/2}y$ if n is odd ($n > 0$).

The map $F(0) \to E(1)$ is clearly fully faithful and essentially surjective, but is not a level-wise equivalence as $E(1)$ is not contractible. Thus our desired classes of Segal spaces should have at least the property that the map

$$Map(E(1), W) \to Map(F(0), W)$$

is an equivalence. The question then is whether that suffices or we need to add further restrictions.

As usual before we can further analyze this question we need some theory.

Definition 4.3. Let W be a Segal space. Then a map f is an equivalence if it has as 2-sided inverse i.e. there is a map h such that $fh \simeq id$ and $hf \simeq id$. The space of equivalences is denoted by $W_{hoequiv}$ and is a subspace of W_1. In particular then for any two objects x, y there is a space of equivalences that is a subspace of the mapping space:

$$hoequiv(x, y) \hookrightarrow map(x, y)$$

The space of equivalences is corepresentable.

Lemma 4.4. *There is an equivalence*

$$Map(E(1), W) \simeq W_{hoequiv}$$

In particular the map $W_{hoequiv} \to W_1$ is induced by the inclusion $F(1) \to E(1)$.

Proof. (Sketch) [Re01, Theorem 6.2, Section 11] The key insight is that $E(1)$ has a filtration by simplicial space $E(1)^{(n)}$, which is the smallest subspace that includes the non-degenerate cell $(xy)^{n/2}$ or alternatively $(xy)^{(n-1)/2}x$. For example, $E(1)^{(1)} = F(1)$. Moreover the colimit of these inclusions satisfies

$$E(1) \cong \text{colim}(E(1)^{(1)} \to E(1)^{(2)} \to E(1)^{(3)} \to ...)$$

Thus we have a limit diagram

$$Map(E(1), W) \cong \lim(Map(E(1)^{(1)}, W) \leftarrow Map(E(1)^{(2)}, W) \leftarrow Map(E(1)^{(3)}, W) \leftarrow ...)$$

Now, we make use of following two technical observations:

(1) The diagram stabilizes past $n = 3$.
(2) For $n = 3$ we have an equivalence

$$Map(E(1)^{(3)}, W) \simeq W_{hoequiv}$$

Justifying these two facts requires some amount of combinatorics, but in some sense relates to the fact that we are dealing with $(\infty, 1)$-categories and being a weak equivalence is determined by the existence of a certain 3-cell. □

We can now use this observation to characterize the locality in several ways:

Proposition 4.5. *(some parts are* [Re01, Proposition 6.4] *and some not) Let W be a Segal space. The following are equivalent:*

(1) The map of spaces

$$Map(E(1), W) \xrightarrow{\simeq} Map(F(0), W) = W_0$$

induced by the map $F(0) \to E(1)$ is an equivalence.

(2) For two objects x, y in W_0 there is an equivalence

$$Path(x, y) \simeq hoequiv(x, y)$$

(3) The following is a homotopy pullback square

$$\begin{array}{ccc} W_0 & \longrightarrow & W_3 \\ \downarrow & & \downarrow{\scriptstyle 0213^*} \\ W_1 & \xrightarrow{(id_x, f: x \to y, id_y)} & W_1 \times_{W_0} W_1 \times_{W_0} W_1 \end{array}$$

here the pullback in the bottom right corner is given by $Map(Z(3), W)$, where $Z(3)$ is the simplicial space that can be depicted as

$$\bullet \to \bullet \leftarrow \bullet \to \bullet$$

(4) The following is a homotopy pullback square

$$\begin{array}{ccc} W_0 & \longrightarrow & W_3 \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (02^*, 13^*)} \\ W_0 \times W_0 & \longrightarrow & W_1 \times W_1 \end{array}$$

Proof. *(1) $\Leftrightarrow$ (2)* Notice we have following diagram

$$\begin{array}{ccc} W_0^{\Delta^1} & \longrightarrow & W_{hoequiv} \\ & \searrow\!\!\!\!\searrow \quad \swarrow\!\!\!\!\swarrow & \\ & W_0 \times W_0 & \end{array}$$

The map on the top is an equivalence if and only if it is a fiber-wise equivalence over $W_0 \times W_0$. But the fiber over a given point x, y in $W_0 \times W_0$ is just

$$Path(x, y) \simeq hoequiv(x, y)$$

giving us the desired equivalence.

(3) $\Leftrightarrow$ (4) We have following diagram

$$\begin{array}{ccc} W_0 & \longrightarrow & W_3 \\ \downarrow & & \downarrow \\ W_1 & \longrightarrow & W_1 \times_{W_0} W_1 \times_{W_0} W_1 \\ \downarrow & & \downarrow \\ W_0 \times W_0 & \longrightarrow & W_1 \times W_1 \end{array}$$

The bottom square is pullback square and so the top square is a pullback square if and only if the rectangle is a pullback square.

(1) $\Leftrightarrow$ *(3)* Notice we have following pullback square

$$\begin{array}{ccc} W_{hoequiv} & \longrightarrow & W_3 \\ \downarrow & \ulcorner & \downarrow \\ W_1 & \longrightarrow & W_1 \times_{W_0} W_1 \times_{W_0} W_1 \end{array}$$

This immediately follows from the fact that a point f in W_1 which has image (id_x, f, id_y) in $W_1 \times_{W_0} W_1 \times_{W_0} W_1$ lifts to W_3 if and only if f is an equivalence. Thus the square above is a homotopy pullback square if and only if the map $W_0 \to W_{hoequiv}$ is an equivalence. ☐

We have done all this stuff, but still haven't really proven that level-wise equivalences and fully faithful and essentially surjective agree.

Theorem 4.6. *Let W, V be a Segal spaces, which satisfy one of the equivalent conditions given above. Then a map $f : W \to V$ is a level-wise equivalence if and only if it is Dwyer-Kan equivalence.*

Proof. If it is a level-wise equivalence then clearly it is fully faithful and essentially surjective by the same argument we have given before. On the other side let us assume it is fully faithful and essentially surjective.

First we prove that the induced map

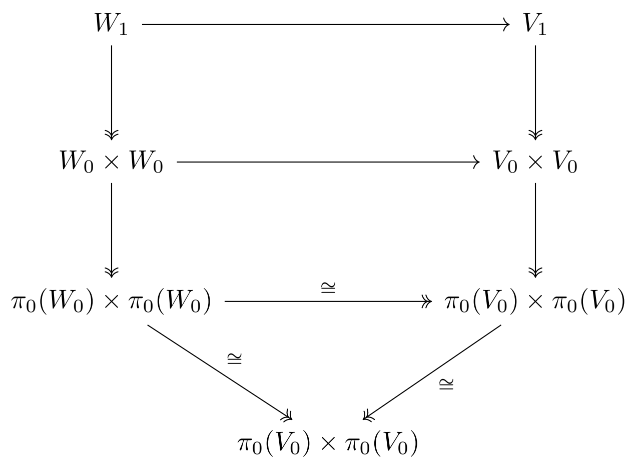
$$\pi_0(W_0) \to \pi_0(V_0)$$

is a bijection of sets. The essentially surjective part implies this map is surjective as every object in V is connected via an equivalence to an object in the image of f. But equivalences are just paths and so this means every point in V_0 is path connected to a point in the image of f. Thus we need to show it is injective.

Let us take two objects x, y in W such that fx and fy are in the same path component. This means there exists an equivalence $fx \to fy$ in the Segal space V. Now we can use the fact that f

is fully faithful to deduce that the corresponding map $x \to y$ is an equivalence as well. This proves $\pi_0(f) : \pi_0(W_0) \to \pi_0(V_0)$ is a bijection.

We now have following diagram:

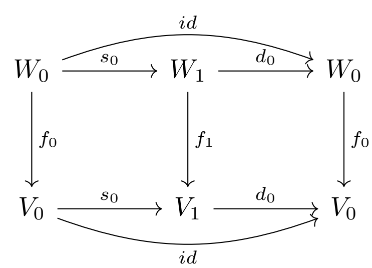


as the maps on both sides are Kan fibrations, the map $W_1 \to V_1$ is an equivalence if and only if it is a fiber-wise equivalence, however, a fiber over a given point (fx, fy) is the map

$$map_W(x, y) \to map_V(fx, fy)$$

which is an equivalence as the map is fully faithful. This proves that $f_1 : W_1 \to V_1$ is a Kan equivalence.

Now we have following diagram:



which means f_0 is a retract of f_1 and thus also an equivalence.

Finally, we have a commutative diagram

$$\begin{array}{ccc} W_n & \xrightarrow{f_n} & V_n \\ \downarrow \simeq & & \downarrow \simeq \\ W_1 \times_{W_0} \ldots \times_{W_0} W_1 & \xrightarrow{\simeq} & V_1 \times_{V_0} \ldots \times_{V_0} V_1 \end{array}$$

where the vertical maps are equivalences by the Segal condition and the bottom horizontal map is an equivalence by the argument above. □

This the result we wanted!

Let's recap what happened. We defined Segal spaces and observed that they don't respect Dwyer-Kan equivalences. We then noticed that if it respects Dwyer-Kan equivalences then it must in particular treat the map $F(0) \to E(1)$ like an equivalence. Finally we proved that this necessary condition is actually sufficient and so we are done.

We are finally in the position to give the desired definition.

Definition 4.7. A Segal space W is called *complete* if the map

$$Map(E(1), W) \to Map(F(0), W_0)$$

induced by the inclusion $F(0) \to E(1)$ is an equivalence.

Remark 4.8. Why is it called complete? If W is a Segal space then the map

$$W_0 \to W_{hoequiv}$$

is an inclusion of spaces and an equivalence if and only if W is complete. Thus W_0 is "complete" in the sense that it has all the information of $W_{hoequiv}$.

Remark 4.9. Notice the key input we needed in the proof was that for a Segal space W we need

$$\pi_0(W_0) \cong Ob(W)/\sim$$

where $Ob(W)/\sim$ is the set of objects up to equivalence. In a general Segal space there is a map of sets

$$\pi_0(W) \to Ob(W)/\sim$$

that is actually a surjection, but might not be injective i.e. in general if two objects are equivalent in the Segal space, they might not be connected by a path in the space W_0. Completeness implies the injectivity of this map, which then implies bijectivity.

Let us end this section on a small note on completeness vs. univalence. In homotopy type theory we study *univalent universes*. A universe $\mathcal{U}$ is called univalent if for two types $A, B : \mathcal{U}$ the map

$$\text{idtoequiv} : A = B \to A \simeq B$$

is an equivalence. How can we make sense of it from the perspective of completeness?

We should think of the universe as really being the 0-level of some complete Segal space (or type maybe?). The identity type $A = B$ corresponds to the space of paths, $Path(A, B)$ and $A \simeq B$ is just the equivalences $hoequiv(A, B)$. Thus univalence is just asking that the map

$$Path(A, B) \to hoequiv(A, B)$$

is an equivalence. But this is one of the equivalent conditions of completenes. Thus there is a direct correspondence between univalence as studied by homotopy type theorists and completeness as studied by higher category theorists.

Stuff About Left Fibrations

Let's change tune and talk about fibrations of $(\infty, 1)$-categories. From this point on we will restrict our attention to quasi-categories and complete Segal spaces. Thus for me an ∞-category will be one of those.

Let $\mathcal{C}$ be a quasi-category. Let x, y, z be three objects. Then we can define a Kan complex $map_{\mathcal{C}}(x, y, z)$ as the Kan complex

$$\begin{array}{ccc} map_{\mathcal{C}}(x,y,z) & \longrightarrow & \mathcal{C}^{\Delta^3} \\ \downarrow & \ulcorner & \downarrow \\ * & \xrightarrow{(x,y,z)} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{array}$$

It is the space of diagrams of the form

$$\begin{array}{ccccc} & & y & & \\ & \nearrow & & \searrow & \\ x & & \longrightarrow & & z \end{array}$$

Now this gives us following diagram

$$\begin{array}{ccccc} Comp_{\mathcal{C}}(f,g) & \longrightarrow & map_{\mathcal{C}}(x,y,z) & \xrightarrow{d_1} & map_{\mathcal{C}}(x,z) \\ \downarrow \simeq & & \simeq \downarrow\!\!\uparrow \simeq & & \\ * & \xrightarrow{(f,g)} & map_{\mathcal{C}}(x,y) \times map_{\mathcal{C}}(y,z) & & \end{array}$$

The vertical map is a trivial Kan fibration. So, for two morphisms $f : x \to y$, $g : y \to z$ the composition can be characterized as a choice of lift of f, g to $map_{\mathcal{C}}(x, y, z)$ along with a projection d_1 to $map_{\mathcal{C}}(x, z)$. So, why does this observation matter?

Fix an object x in $\mathcal{C}$. We want to define a functor

$$map_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}$$

It is immediately obvious that the image of a point y is the space $map_{\mathcal{C}}(x, y)$. What is the image of a map $f : y \to z$? It should be a map

$$f_* : map_{\mathcal{C}}(x, y) \to map_{\mathcal{C}}(x, z)$$

that takes a point g to the point fg. The problem, however, is that such a composition does not really exist. Rather we have to make a choice of lift as described in the diagram above. Theoretically we could choose a lift, but a random choice of compositions would hardly be functorial.

This means we need to take a very different approach to this problem:

(1) Give up on the Yoneda lemma and all its benefits
(2) Try to construct the Yoneda lemma using a different method

5.1 The Classical Story. As we are all quite fond of the Yoneda lemma I think it's best to take the second route! Here we use following result from classical category theory:

Definition 5.1. Let $\mathcal{C}$ be a 1-category. A *Grothendieck fibration* is a functor $p : \mathcal{D} \to \mathcal{C}$ such that for each morphism $f : c_1 \to c_2$ and lift d_2 of c_2 there is a unique lift of f.

$$\begin{array}{ccc} d_1 & \overset{\exists!\hat{f}}{\dashrightarrow} & d_2 \\ \Downarrow p & & \Downarrow p \\ c_1 & \xrightarrow{f} & c_2 \end{array}$$

Notice the uniqueness implies in particular that the fiber of a Grothendieck fibration is necessarily a set.

We will denote the full subcategory of $\mathcal{C}at_{/\mathcal{C}}$ consisting of Grothendieck fibrations by $\mathcal{G}roth_{/\mathcal{C}}$. We have following theorem about this category.

Theorem 5.2. *There is an equivalence of categories*

$$\mathcal{G}roth_{/\mathcal{C}} \underset{\int_{\mathcal{C}}}{\overset{\mathcal{F}ib}{\rightleftarrows}} Fun(\mathcal{C}^{op}, \mathcal{S}et)$$

Proof. We will construct functors in both directions. First, for a given functor $F : \mathcal{C}^{op} \to \mathcal{S}\text{et}$ let $\pi : \int_{\mathcal{C}} F \to \mathcal{C}$ be defined by

(1) Objects: tuples $(c \in \mathcal{C}, x \in F(c))$
(2) Morphisms: A map $f : (c, x) \to (d, y)$ is a map $f : c \to d$ such that $F(f)(y) = x$.

it comes with an evident projection functor to $\mathcal{C}$.

On the other hand for a given fibration $p : \mathcal{D} \to \mathcal{C}$ define $\mathcal{F}\text{ib}(p) : \mathcal{C}^{op} \to \mathcal{S}\text{et}$ as the functor that takes an object c to the fiber $\mathcal{F}\text{ib}(p)(c) = p^{-1}(c)$ and for a given morphism $f : c \to d$ in $\mathcal{C}$ and given object y in $p^{-1}(d)$ defines the image of y under the map $\mathcal{F}\text{ib}(p)(f)$ as the domain of the unique lift of f along y.

$$\begin{array}{ccc} \mathcal{F}\mathrm{ib}(p)(f)(y) & \overset{\exists ! \hat{f}}{\dashrightarrow} & y \\ \Downarrow p & & \Downarrow p \\ c & \xrightarrow{f} & d \end{array}$$

It is then a formality to check that these two constructions are functorial and inverses. □

Thus every set valued presheaf can be seen also as a certain fibration. This is the view we want to take. For that we need to generalize the definition given above.

5.2 Left Fibrations. In order to find the appropriate generalization we will apply the following slogan:

Uniqueness in set theory = Contractibility in homotopy theory!

So, we want a definition where for a given morphism in the base and lift of the codomain there is a *contractible* choice of lifts! How can we make this more precise?

Before we move on some comments about references. Fibrations of ∞-categories where first studeid by the father of quasi-category theory, Joyal [Jo08] [Jo09], using quasi-categories. It was then picked up and studied in much more detail by Lurie [Lu09], again using quasi-categories.

The complete Segal approach to fibrations is originally due to Charles Rezk, however, he never wrote a single word about this. The first paper that studies some aspects of fibrations of Segal spaces is due to de Brito [dB16]. Here, however, we want fibrations of simplicial spaces which as far as I know can only be found in [Ra17].

Remark 5.3. It is more convenient to everything to the covariant version and hence we will do so!

If $p : \mathcal{D} \to \mathcal{C}$ is an ∞-category (whatever model), then $J(\mathcal{C})$ is the *space of objects* and $J(\mathcal{C}^{[1]})$ is the *space of morphisms*. The functor p gives us maps

$$J(p) : J(\mathcal{D}) \to J(\mathcal{C})$$

$$J(p^{[1]}) = J(\mathcal{D}^{[1]}) \to J(\mathcal{C}^{[1]})$$

We can put these spaces into the following diagram:

$$\begin{array}{ccc} J(\mathcal{D}^{[1]}) & \xrightarrow{s} & J(\mathcal{D}) \\ \downarrow & & \downarrow \\ J(\mathcal{C}^{[1]}) & \xrightarrow{s} & J(\mathcal{C}) \end{array}$$

where s takes a morphism to its source.

A point in the pullback of this diagram is exactly the data of a morphism in $\mathcal{C}$ along with a lift of the source. But this should be equivalent to having a morphism in $\mathcal{D}$.

Thus our desired fibration should exactly have the property that this commutative square is a homotopy pullback square.

Definition 5.4. *(Close enough definition)* Let $p : \mathcal{D} \to \mathcal{C}$ be a map of ∞-categories. Then we say p is a *left fibration* if the square above is a pullback square of spaces.

This definition has the right homotopy type, but lacks the right fibrational properties to be useful for us. We cannot use this definition to construct model structures. Thus we need one last step here in order to get a working definition.

Definition 5.5. Let $p : \mathcal{D} \to \mathcal{C}$ be a fibration of quasi-categories or complete Segal spaces. Then p is a *left fibration* if the following is a homotopy pullback square of spaces:

$$\begin{array}{ccc} J(\mathcal{D}^{[1]}) & \xrightarrow{s} & J(\mathcal{D}) \\ \downarrow & & \downarrow \\ J(\mathcal{C}^{\Delta^1}) & \xrightarrow{s} & J(\mathcal{C}) \end{array}$$

This definition works for ∞-categories. The goal is now to generalize our perspective to non-fibrant objects (i.e. not ∞-categories) to then build a model structure. Here we have to go model dependent again.

Let us start with complete Segal spaces, because here it is much easier.

Definition 5.6. Let $p : Y \to X$ be a map of simplicial spaces (really no assumptions on them!). Then p is a left fibration if:

(1) p is a Reedy fibration.
(2) For each n the following is a homotopy pullback square

$$\begin{array}{ccc} Y_n & \xrightarrow{p_n} & X_n \\ {\scriptstyle 0^*}\downarrow & \ulcorner & \downarrow{\scriptstyle 0^*} \\ Y_0 & \xrightarrow[p_0]{} & X_0 \end{array}$$

where 0^* is induced by the unique map $0 : [0] \to [n]$ takes the point to 0.

This definition is a proper generalization of the definition above. In fact we have following lemma:

Lemma 5.7. *Let $p : V \to W$ be a Reedy fibration of Segal spaces. Then p is a right fibration if and only if the square*

$$\begin{array}{ccc} V_1 & \xrightarrow{p_1} & W_1 \\ {\scriptstyle 0^*}\downarrow & & \downarrow{\scriptstyle 0^*} \\ V_0 & \xrightarrow[p_0]{} & W_0 \end{array}$$

is a homotopy pullback square.

Proof. One side is trivial, for the other side notice we have the diagram:

$$\begin{array}{ccc} V_n & \longrightarrow & W_n \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ V_{n-1} \times_{V_0} V_1 & \longrightarrow & W_{n-1} \times_{W_0} W_1 \\ \downarrow{\scriptstyle (0^*,0^*)} & & \downarrow{\scriptstyle (0^*,0^*)} \\ V_0 \times_{V_0} V_0 & \longrightarrow & W_0 \times_{W_0} W_0 \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ V_0 & \longrightarrow & W_0 \end{array}$$

The top square is a pullback square by the Segal condition. The bottom square is trivially a pullback square. This implies that if the middle square is a pullback square then the whole rectangle is a pullback square. The desired result now follows from induction starting with $n = 2$. □

The goal is to study the homotopy theory of such left fibrations by constructing a model structure. Before we go there, let us analyze some simple examples.

Example 5.8. Let $p : L \to F(0)$ be a left fibration. What can we say about R? First of all it is Reedy fibrant as it is a Reedy fibration over the final object. Moreover, we have following pullback square:

$$\begin{array}{ccc} L_n & \longrightarrow & L_0 \\ \downarrow & & \downarrow \\ F(0)_n & \longrightarrow & F(0)_0 \end{array}$$

But $F(0)$ is the simplicial space that at each level is just the point, thus this being a pullback square means $L_n \to L_0$ is an equivalence. Thus $L \to F(0)$ is a left fibration if (and only if) L is a *homotopically constant* simplicial space, meaning the simplicial maps are all equivalences. Thus the data of L is (up to equivalence) fully determined by L_0, a space! This matches with our intuition that it should corespond to a functor out of $F(0)$ into spaces and that is exactly just a choice of a space.

Example 5.9. Let $p : L \to F(1)$ be a left fibration. What can we say about L this time? Let us first break down the combinatorics of $F(1)$. It is a discrete simplicial space, where

$$F(1)_n = Hom_{\Delta}([n],[1])$$

we can characterize this set as an increasing chain of length $n+1$ consisting of $0, 1$. Thus we roughly get

$$F(1) = \{0,1\} \leftleftarrows \{00, 01, 11\} \leftleftarrows \{000, 001, 011, 111\} \leftleftarrows \cdots$$

the face maps will drop one of the digits and the degeneracy maps will simply repeat a certain digit. In particular then $F(1)_0$ has two non-degenerate cells, $0, 1$, and $F(1)_1$ has one non-degenerate cell 01 and two degenerate ones $00, 11$.

This means that the simplicial space L takes the following form

$$L = L_{/0} \coprod L_{/1} \leftleftarrows L_{/00} \coprod L_{/01} \coprod L_{/11} \leftleftarrows L_{/000} \coprod L_{/001} \coprod L_{/011} \coprod L_{/111} \leftleftarrows \cdots$$

where $L_{/ijk}$ denotes the fiber of L over the point ijk.

The Reedy fibrancy condition implies that all these are actually Kan complexes. What does the other left fibration condition tell us? For $n = 1$ we get the pullback square:

$$\begin{array}{ccc} L_{/00} \coprod L_{/01} \coprod L_{/11} & \xrightarrow{s} & L_{/0} \coprod L_{/1} \\ \downarrow & \lrcorner & \downarrow \\ \{00, 01, 11\} & \xrightarrow[s]{} & \{0, 1\} \end{array}$$

however, the source map will take $00 \mapsto 0$, $01 \mapsto 0$, $11 \mapsto 1$ and so just pulling back the diagram means that we the equivalence

$$L_{/00} \coprod L_{/01} \coprod L_{/11} \xrightarrow{\simeq} L_{/0} \coprod L_{/0} \coprod L_{/1}$$

Thus in particular we have equivalences

$$L_{/00} \to L_{/0}$$
$$L_{/01} \to L_{/0}$$
$$L_{/11} \to L_{/1}$$

this generalizes in the appropriate way. In particular the left fibration condition implies that the fiber over the point $11...1 \in F(1)_n$ is equivalent to L_1 and the fiber over any other point is just equivalent to L_0.

How does this relate to our functor perspective? Well the essential data of this simplicial space can be reduced to the zig-zag

$$L_{/0} \xleftarrow[\simeq]{} L_{/01} \longrightarrow L_{/1}$$

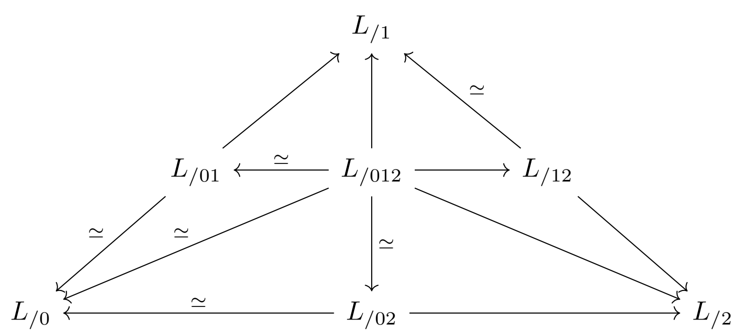
or more specifically a map of Kan complexes

$$L_{/01} \to L_{/1}$$

which is exactly the data of a functor from the free arrow valued in spaces.

Remark 5.10. Notice how the notion of functor automatically incorporates a "zig-zag" of equivalences, which matches the diagrams we observed for mapping spaces.

The data of a right fibration over $F(2)$ can be depicted by the diagram of spaces:



which has essential data:

$$L_{/012} \to L_{/12} \to L_{/2}$$

This previous example can be appropriately generalized to a left fibration over $F(n)$. In particular, we can then observe that its data reduces to chain of spaces

$$L_{/01...n} \to L_{/1...n} \to ... \to L_{/(n-1)n} \to L_{/n}$$

where these spaces are fibers over the appropriate points in $F(n)_k$.

Let us now move on to the main topic: a homotopy theory of left fibrations.

Theorem 5.11. *Let X be a fixed simplicial space. There is a left proper, cofibrantly generated, simplicial model category on the over-category $s\mathcal{S}_{/X}$ such that*

1. *the fibrant objects are left fibration.*
2. *the cofibrations are monomorphisms*
3. *A map $f : Y \to Z$ over X is an equivalence if for every left fibration $L \to X$ the induced map*
$$f^* : Map_{/X}(Z, L) \to Map_{/X}(Y, L)$$
is a Kan equivalence.
4. *A map between left fibrations $L_1 \to L_2$ is a weak equivalence (fibration) if it is a Reedy weak equivalence (Reedy fibration).*

This model structure is called the covariant model structure.

Proof. As usual this is just a Bousfield localization. We localize the Reedy model structure on $s\mathcal{S}_{/X}$ with respect to the set of maps

$$\mathcal{L} = \{F(0) \xrightarrow{0} F(n) \to X\}$$

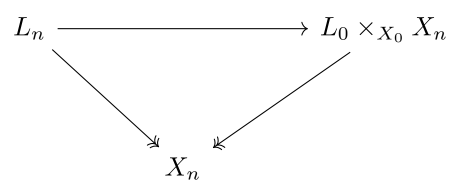
The only property we need to actually check is that fibrant objects in that localization model structure are left fibrations. As soon as we have that everything else follows from the formal properties of Bousfield localizations.

Concretely, we need to prove the following statement: Let $L \to X$ be a Reedy fibration of simplicial spaces. Then it is a left fibration if and only if for every map $F(0) \to F(n)$ over X the induced map

$$Map_{/X}(F(n), L) \to Map_{/X}(F(0), L)$$

is a Kan equivalence.

Here we use the classic "fiber-wise criterion for Kan equivalences". Concretely, we have the diagram:



The top map being an equivalence is exactly the criterion for being a left fibration. As the two legs are Kan fibrations this is equivalent to being a fiber-wise Kan equivalence i.e.

$$L_n \times_{X_n} * \to L_0 \times_{X_0} *$$

being an equivalence of spaces.

Now we just observe that

$$Map_{/X}(F(n), L) = Map(F(n), L) \times_{Map(F(n),X)} * \cong L_n \times_{X_n} *$$

and similarly

$$Map_{/X}(F(0), L) = Map(F(0), L) \times_{Map(F(0),X)} * \cong L_0 \times_{X_0} *$$

giving us the desired result. □

This gives us a working homotopy theory of left fibrations for simplicial spaces. How about simplicial sets? When we want to give a definition of a left fibration for simplicial sets, we cannot use our intuition about unique lifts, as each level of a simplicial set is only a set. We thus need an alternative approach.

Here is something that can help us. As we saw before, we have two Quillen equivalences between quasi-categories and complete Segal spaces, namely the adjunctions $(t_!, t^!)$ and (p_1^*, i_1^*). Thus what we would expect is that they induce equivalences of left fibrations.

Concretely, if $T \to S$ is a map of simplicial sets that is a left fibration then we would expect $t^!T \to t^!S$ be a left fibration of simplicial spaces. In particular this means we have lifting diagrams

$$\begin{array}{ccc} F(0) \times \Delta^l \coprod_{F(0)\times\partial\Delta^l} F(n) \times \partial\Delta^l & \longrightarrow & t^!T \\ \downarrow & \nearrow & \downarrow \\ F(n) \times \Delta^l & \longrightarrow & t^!S \end{array}$$

which by adjunction and the fact that $t_!(F(n) \times \Delta^l) = \Delta^n \times I[l]$ is equivalent to

$$\begin{array}{ccc} F(0) \times \Delta^l \coprod_{F(0)\times\partial\Delta^l} F(n) \times \partial\Delta^l & \longrightarrow & t^!T \\ \downarrow & \nearrow & \downarrow \\ F(n) \times \Delta^l & \longrightarrow & t^!S \end{array}$$

If we take $n = l = 1$ then the shape we get is a square where two are sides are invertible and we can lift along the shape that removes one invertible side. Collapsing the constant invertible side exactly tells us that we have a lift along a left horn. We can use this as a guide for following definition.

Definition 5.12. A map of simplicial sets $T \to S$ is a *left fibration* if it lifts along all left horns.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & T \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & S \end{array}$$

where $0 \leq i < n$.

Again, we want a homotopy theory for left fibration of simplicial sets. However, this time we cannot use the theory of Bousfield localizations as simplicial sets don't come with a built-in model structure.

Instead we have to use an appropriate result that constructs a model structure with the desired properties.

Proposition 5.13 ([Lu09, Proposition 2.1.4.7])**.** *Let S be a simplicial set. There is a left proper combinatorial model structure on $s\mathcal{S}et_{/S}$ such that*

(1) Cofibrations are monomorphisms
(2) Fibrant objects are left fibrations.

This model structure is also called the covariant model structure.

The key input is [Lu09, Proposition A.2.6.15], which proves the existence of a combinatorial model structure using generating cofibrations.

The two notions of left fibations actually do compare appropriately.

Proposition 5.14. *Let X be a simplicial space. Then*

$$s\mathcal{S}et_{/i_1^*X} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} s\mathcal{S}_{/X}$$

is a Quillen adjunction, where both sides have the covariant model structure.

Proof. Clearly the left adjoint preserves cofibrations so we only need to show the right adjoint takes trivial fibrations between fibrant objects to trivial fibrations. But trivial fibrations between fibrant objects are just trivial Reedy fibrations which we already know are preserved. Thus we only need to prove that the i_1^* preserves right fibrations.

Again we already know it preserves Reedy fibrations, so we only have to check it preserves the locality condition. By adjunction we just have to prove that

$$p_1^*\Lambda^n_n \to p_1^*\Delta^n$$

is a trivial cofibration in the covariant model structure on $s\mathcal{S}_{/X}$.

For that we use some combinatorics to assemble right horns out of maps $F(n-1) \to F(n)$. □

Similarly, we have:

Proposition 5.15. *Let X be a simplicial space. Then*

$$s\mathcal{S}_{/X} \underset{t^!}{\overset{t_!}{\rightleftarrows}} s\mathcal{S}et_{/t_!X}$$

is a Quillen adjunction, where both sides have the covariant model structure.

In fact a very similar argument to the absolute case proves that both these Quillen adjunctions are Quillen equivalences. Indeed, it involves the same three steps:

(1) Prove (p_1^*, i_1^*) is a Quillen equivalence.
(2) Prove the composition is a Quillen equivalence.
(3) Deduce $(t_!, t^!)$ is a Quillen equivalence as well

Remark 5.16. The fact that we have different definitions of a right fibrations for complete Segal spaces and quasi-categories and need prove they are equivalent is a good example why it is valuable to have a "model independent" approach that gives a treatment of right fibrations that doesn't depend on a particular model of $(\infty, 1)$-category.

This is exactly the point of the work of Riehl and Verity [RV18], which gives a treatment of right fibrations from an ∞-cosmos perspective.

5.3 Right Fibrations. Where there is a left there should be a right! Concretely, we think of left fibrations are classifying covariant functors. Thus we should have a notion of right fibration that classifies contravariant functors. Here we only give basic definitions and leave it to the interested reader to fill the analogous results.

Definition 5.17. A Reedy fibration of simplicial spaces $R \to X$ is a *right fibration* if

$$\begin{array}{ccc} R_n & \longrightarrow & X_n \\ \downarrow{\scriptstyle n^*} & \lrcorner & \downarrow{\scriptstyle n^*} \\ R_0 & \longrightarrow & X_0 \end{array}$$

is a pullback square.

Similarly a map of simplicial sets $R \to S$ is a right fibration if it lifts again right horns $\Lambda^n_i \to \Delta^n$, $0 < i \leq n$

Theorem 5.18. *For a given simplicial space X and simplicial set S there are nice model structures on $s\mathcal{S}_{/X}$ and $s\mathcal{S}et_{/S}$ where the fibrant objects are right fibrations and are known as the contravariant model structure.*

Moreover, there are Quillen equivalences

$$s\mathcal{S}et_{/i_1^*X} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} s\mathcal{S}_{/X} \qquad s\mathcal{S}_{/X} \underset{t^!}{\overset{t_!}{\rightleftarrows}} s\mathcal{S}et_{/t_!X}$$

Something about Straightening

In the last section we discussed how we want to use fibrations instead of functors in the ∞-categorical setting. We then introduced an ∞-categorical version of a fibration (left fibrations). We now have to prove that our new notion of fibration are equivalent to funtors.

It turns out proving this is extremely complicated. It was first proven by Lurie in [Lu09], where it takes up most of Section 2 of his book. Concretely, Lurie proves the following result:

Theorem 6.1 ([Lu09, Theorem 2.2.1.2])**.** *Let S be a simplicial set. There is a Quillen equivalence:*

$$s\mathcal{S}et_{/S} \underset{\mathrm{Un}_S}{\overset{\mathrm{St}_S}{\rightleftarrows}} \mathcal{S}^{\mathfrak{C}[S]}$$

where the left hand side has the covariant model structure over S and the right hand side has the projective model structure.

The St and Un are meant to be abbreviations of “straightening”, “unstraightening”, somehow suggesting that fibration is an unstraightened object, whereas a functor is straightened.

Fortunately, since the book was published others have now written easier proofs. For example, there is a proof by Stevenson [St15]. A nice summary of that proof can be found in the lecture notes by Rune Haugseng [Ha17, Chapter 8].

There is a further proof by Heuts and Moerdijk, which goes in two steps. First they prove the result for the case when base is a 1-category, [HM15], and then they generalize it to the case of a general simplicial category [HM16]. Here we will review the arguments given in [HM15] to prove the straightening construction for categories.

Throughout A will be a fixed small category. We will prove following theorem:

Theorem 6.2 ([HM15, Theorem C])**.** *There are two Quillen equivalences*

$$s\mathcal{S}et^A \underset{h^*}{\overset{h_!}{\rightleftarrows}} s\mathcal{S}et_{/NA} \underset{r^*}{\overset{r_!}{\rightleftarrows}} s\mathcal{S}et^A$$

between the projective and covariant model structure.

Remark 6.3. One interesting aspect of this proof is that they simplify things by defining an inverse. This is not true in the proof by Lurie.

In order to get there we will first prove a Yoneda lemma for left fibrations.

6.1 Yoneda Lemma for Left Fibrations. Here is a classic result in category theory:

Theorem 6.4. *Let $\mathcal{C}$ be a category and c an object. For any functor $F : \mathcal{C} \to \mathcal{S}et$ there is an isomorphism*

$$Map(Hom_{\mathcal{C}}(c, -), F) \cong F(c)$$

In order to prove this statement we construct two maps in both directions:

- On the one side we take a natural transformation α to the element $\alpha_c(id_c) \in F(c)$.
- On the other side we take an element $x \in F(c)$ to the natural transformation that takes a morphism $f : c \to d$ to $F(f)(x) \in F(d)$.

Then we just prove these maps are inverses.

However, we claimed that everything about functors can be translated into fibrations. So, what is the fibrational analogue to the Yoneda lemma? First, observe that $\int_{\mathcal{C}} Hom_{\mathcal{C}}(c, -) = \mathcal{C}_{c/}$ (exercise!). That means the fibrational Yoneda lemma would be

Theorem 6.5. *Let $\mathcal{C}$ be a category and c an object. For any Grothendieck fibration $p : \mathcal{D} \to \mathcal{C}$ there is an isomorphism:*

$$Fun_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{D}) \cong p^{-1}(c)$$

The proof is actually quite analogous. We take a map $\alpha : \mathcal{C}_{c/} \to \mathcal{D}$ to its value at id_c, $\alpha(id_c) \in \mathcal{D}$ and then prove this is a bijection.

We want to prove the analogous statement for left fibrations. First here is a lemma:

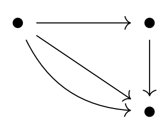
Lemma 6.6. *Let $\mathcal{C}$ be an ∞-category and c an object. Then the under-category defined as*
$$\mathcal{C}_{c/} = \mathcal{C}^{[1]} \times_{/C} *$$
gives us a left fibration $\pi_c : \mathcal{C}_{c/} \to \mathcal{C}$.

Proof. We need to observe that the square

$$\begin{array}{ccc} J(\mathcal{C}_{c/}^{\Delta^1}) & \longrightarrow & J(\mathcal{C}^{\Delta^1}) \\ \downarrow & & \downarrow \\ J(\mathcal{C}_{c/}) & \longrightarrow & J(\mathcal{C}) \end{array}$$

is a pullback square of spaces. However, by definition the pullback of the diagram is given by
$$J(\mathcal{C}^{\Delta^1}) \times_{J(\mathcal{C})} J(\mathcal{C}_{c/}) \simeq Map_{c/}(\Delta^1 \coprod_{\Delta^0} \Delta^1, \mathcal{C})$$
i.e. maps from $\Delta^1 \coprod_{\Delta^0} \Delta^1 \to \mathcal{C}$ that take the initial point to c. By the Segal condition, this is equivalent to $Map_{c/}(\Delta^2, J(\mathcal{C}))$.

On the other side for $J((\mathcal{C}_{c/})^{\Delta^1})$ we have
$$J((\mathcal{C}_{c/})^{\Delta^1}) \cong Map(\Delta^0, J((\mathcal{C}_{c/})^{\Delta^1})) \simeq Map(\Delta^1, \mathcal{C}_{c/}) \simeq Map((\Delta^1 \times \Delta^1) \coprod_{\Delta^1} \Delta^0, \mathcal{C})$$
where $(\Delta^1 \times \Delta^1) \coprod_{\Delta^1} \Delta^0$ is depicted as



The desired result now follows from the fact that the inclusion

is a categorical equivalence. □

Remark 6.7. In fact this holds even for a Segal space (completeness is not required).

The Yoneda Lemma then can be phrased as:

Theorem 6.8. *Let $\mathcal{C}$ be an ∞-category and c an object. Moreover, let $p : \mathcal{L} \to \mathcal{C}$ be a left fibration. Then we have an equivalence*

$$Map_{\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{L}) \xrightarrow{\simeq} p^{-1}(c)$$

given by evaluation at the identity.

Notice $p^{-1}(c) = Map_{/C}(\Delta^0, \mathcal{L})$. Thus the statement above is equivalent to the map

$$Map_{\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{L}) \xrightarrow{\simeq} Map_{\mathcal{C}}(\{c\}, \mathcal{L})$$

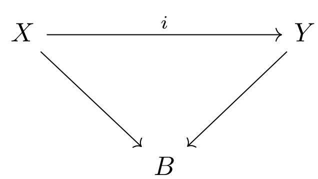
induced by the map $\Delta^0 \to \mathcal{C}_{c/}$ is an equivalence for every left fibration $\mathcal{L}$.

Howevever, the covariant model structure is a simplicial model structure. So this can then just be rephrased as:

Lemma 6.9 ([HM15, Lemma 2.4])**.** *Let $\mathcal{C}$ be an ∞-category and c an object. Then $\Delta^0 \to \mathcal{C}_{c/}$ is a covariant equivalence!*

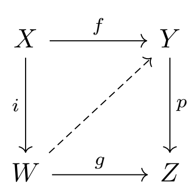
The key to prove this statement is actually an interesting observation by Heuts and Moerdijk that holds for all simplicial sets.

Lemma 6.10 ([HM15, Lemma 2.1])**.** *Consider a monomorphism i of simplicial sets over B*



Now suppose there exist a retraction $r : Y \to X$ such that $ri = id_X$ and a homotopy $h : \Delta^1 \times Y \to Y$ (relative to X) ir to id_Y. Then i is a trivial cofibration in the covariant model structure.

Proof. We need to prove that the diagram



has a lift, where p is a fibration in the covariant model structure. The point is to use the retract to translate the diagram and use our knowledge from classical homotopy theory.

We can factor the square as follows:

$$\begin{array}{ccccc} X & \xrightarrow{id_X \times \{1\}} & \Delta^1 \times X \coprod_{\{0\} \times X} \{0\} \times Y & \xrightarrow{f\pi_X \cup fr} & W \\ \big\downarrow & & \big\downarrow & \overset{k}{\nearrow} & \big\downarrow p \\ Y & \xhookrightarrow{id_Y \times \{1\}} & \Delta^1 \times Y & \xrightarrow{gh} & Z \end{array}$$

The left hand side is a trivial cofibration in the covariant model structure as it is the pushout product of

$$\{0\} \to \Delta^1$$
$$X \to Y$$

both of which are cofibrations and where the first one is a trivial cofibration. So, a lift exist. The desired lift is now simply given by restricting to Y. □

Remark 6.11. Notice in this statement the direction of the homotopy from ir to id_Y actually matters. Concretely, we can only use the inclusion $\{0\} \hookrightarrow \Delta^1$ and that only works if we have a homotopy $ir \Rightarrow id_Y$. Thus it can be seen as a directed generalization of deformation retracts. Thus from a classical perspective we can see the covariant model structure as simplicial sets over B localized with respect to left directed homotopies.

Moreover, note r and h are not required to be maps over B.

We will now use this lemma to prove the desired Yoneda Lemma, as done in [HM15, Lemma 2.4]:

Proof of Lemma 6.9. By the previous lemma all we need is a map $r : \mathcal{B}_{b/} \to \Delta^0$ along with a homotopy $h : ri \Rightarrow id$. The map r is obvious. Now we describe $h : \Delta^1 \times \mathcal{B}_{b/} \to \mathcal{B}_{b/}$. Let (α, β) be an n-simplex. That means $\alpha : [n] \to [1]$ can be seen as a chain of $n+1$ digits 000...0111...1 and β can be seen as a chain of morphisms

$$b \to b_0 \to b_1 ... \to b_n$$

Let k be the minimum index such that $\alpha(k) = 1$ (if it exists). Then we define $h(\alpha, \beta)$ as the element in $(\mathcal{B}_{b/})_n$ of the form

$$b \xrightarrow{=} b \xrightarrow{=} b \xrightarrow{=} ... \xrightarrow{=} b \to b_k \to ... \to b_n$$

and if it does not we map it to itself.

Notice $h|_{\{0\} \times \mathcal{B}_{b/}} = \{id_b\}$, the constant map at id_b, and $h|_{\{1\} \times \mathcal{B}_{b/}} = id$. Thus $h : ir \Rightarrow id$ satisfies all the required properties and we are done. □

Remark 6.12. Notice one interesting difference between this Yoneda lemma and the classical one for 1-categories. We have proven that there is a trivial Kan fibration

$$Map(\mathcal{B}_{b/}, L) \to Map(\{b\}, L)$$

is a trivial Kan fibration. This implies that there is a section

$$Map(\mathcal{B}_{b/}, L) \to Map(\{b\}, L)$$

However, we don't really know what this section is. In particular, we didn't even know this section existed before we proved this map is an equivalence. So, in particular we cannot actually repeat the classical proof which involves constructing an explicit inverse.

The issue is that an explicit inverse would involve defining a functor out of a given value and that is quite difficult because of all the coherence issues.

We will use the Yoneda lemma in the coming subsections to prove straightening.

6.2 Two Equivalences between Functors and Fibrations. We now move on to construct two Quillen adjunctions, namely:

$$\mathrm{s}\mathcal{S}\mathrm{et}^A \underset{h^*}{\overset{h_!}{\rightleftarrows}} \mathrm{s}\mathcal{S}\mathrm{et}_{/NA}$$

$$\mathrm{s}\mathcal{S}\mathrm{et}_{/NA} \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathrm{s}\mathcal{S}\mathrm{et}^A$$

and prove they are Quillen adjunctions.

First let us start with h, the *homotopy colimit functor*. For a given functor $F : A \to \mathrm{s}\mathcal{S}\mathrm{et}$ we will take the following steps:

(1) First we can think of F as a simplicial object in $\mathrm{Fun}(A, \mathcal{S}\mathrm{et})$
$$F_\bullet : A \to \mathcal{S}\mathrm{et}$$
(2) Now we can apply functor $\int_A$ to this to get a simplicial object in $\mathcal{G}\mathrm{roth}_{/A}$.
$$\int_A F_\bullet \to A$$
(3) Next we can apply the nerve construction N
$$N(\int_A F)_\bullet \to NA$$
(4) Finally, notice $N(\int_A F)$ is a simplicial object in simplicial sets i.e. a bisimplicial set and so we can take the diagonal and notice it comes with a map NA.
(5) We will denote the diagonal of this level-wise Grothendieck construction by $h_!(F) \to NA$.

Very explicitly an n-simplex in $h_!(F)$ is given by a pair
$$(a_0 \to a_1 \to ... \to a_n, x \in F(a_0)_n)$$

We now want to define the right adjoint h^*. Notice that in particular $h_!(Hom(b-,)) = N(A_{b/})$ Thus for a given map of simplicial sets $X \to NA$ we have define $h^*(X) : A \to \mathrm{s}\mathcal{S}\mathrm{et}$ as the functor given by
$$h^*(X)(b) = Map_{NA}(N(A_{b/}), X)$$

We are now ready to study this adjunction:

Lemma 6.13 ([HM15, Lemma 3.1, Lemma 3.3])**.** *The adjunction*

$$s\mathbb{S}et^A \underset{h^*}{\overset{h_!}{\rightleftarrows}} s\mathbb{S}et_{/NA}$$

is a Quillen adjunction between the projective model structure and the covariant model structure.

Proof. By inspection $h_!$ preserves cofibrations (as it preserves all monomorphisms). Moreover, $h_!$ takes a generating trivial cofibration of the projective model structure

$$\Lambda^n_i \times Hom(b, -) \to \Delta^n \times Hom(b, -)$$

to

$$\Lambda^n_i \times N(A_{b/}) \to \Delta^n \times N(A_{b/})$$

which is a trivial cofibration in the covariant model structure as the covariant model structure is simplicial. □

Remark 6.14. Notice, the adjunction given above is simplicially enriched, meaning we have a natural isomorphism of simplicial sets

$$Map_{NA}(h_!F, X) \cong N(Fun(F, h^*X))$$

We now move on to define r, the *rectification functor*. This should be seen as an eventual inverse to $(h_!, h^*)$. It suffices to characterize $r_!(\Delta^n \to NA)$ as every object in $s\mathbb{S}\text{et}_{/NA}$ is a colimit of such objects and $r_!$ is a left adjoint.

A simplex $\alpha : \Delta^n \to NA$ equivalently describes a functor $\alpha : [n] \to A$ and so we can define

$$r_!(A)(\alpha) = N(\alpha \downarrow b)$$

Similarly, we can describe r^* as

$$Hom_{NA}(\alpha : \Delta^n \to NA, r^*(F)) \cong Hom(\alpha, F)$$

We are now ready to observe that this gives us a Quillen adjunction.

Lemma 6.15 ([HM15, Lemma 4.1, Lemma 4.2])**.** *The adjunction*

$$s\mathbb{S}et_{/NA} \underset{r^*}{\overset{r_!}{\rightleftarrows}} s\mathbb{S}et^A$$

is a Quillen adjunction, between the covariant model structure and the projective model structure.

Proof. We will prove that r^* takes fibrations to fibration and trivial fibrations to trivial fibrations. However, both of those are determined via lifting properties against maps into Δ^n. So, we will make following more general observation:

For a given map $G \to F$ in $s\mathbb{S}\text{et}^A$ and subcomplex $S = \partial\Delta^n, \Lambda^n_i$ $i < n$ solving the lifting problem

$$\begin{array}{ccc} S & \longrightarrow & r^*(G) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & r^*(F) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow & \\ & NA & \end{array}$$

is equivalent to solving the lifting problem

$$\begin{array}{ccc} S & \longrightarrow & G(\alpha(n)) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & F(\alpha(n)) \end{array}$$

for every $\alpha : \Delta^n \to NA$. □

Remark 6.16. It is interesting to notice that this really fails for the case Λ^n_n. For example take the first non-trivial case, Λ^1_1.

We have now constructed two Quillen adjunctions. We can now move on to the main goal, namely proving that both are Quillen equivalences. As one would expect the argument is quite long and so we break it down.

We will prove the following two statements:

(1) There exists a natural equivalence $\tau : r_! h_!(F) \to F$, which is a weak equivalence if F is projectively cofibrant.
(2) For any map of simplicial sets $X \to NA$ there exists a natural zigzag

$$X \to LX \leftarrow h_! r_!(X)$$

of covariant weak equivalences over NA.

If we prove these two statements, then this proves that the derived functors of $r_!$ and $h_!$ are mutual inverses and in particular $Lh_! \simeq Rr^*$.

We will now move on to prove these two propositions. The first proposition can be proven directly.

Proposition 6.17. *For any functor $F : A \to s\mathcal{S}et$, there exists a natural map $\tau : r_! h_!(F) \to F$, which is a weak equivalence if F is projectively cofibrant.*

Proof. The proof has several steps:

(Step 1) Existence of τ: We first need to define a natural transformation $\tau : r_! h_!(F) \to F$. First we directly compute that for a given object $b \in A$ we have

$$r_! h_!(F)(b)_n = \{(x \in F(a_0)_n, a_0 \xrightarrow{\alpha_1} ... \xrightarrow{\alpha_n} a_n \xrightarrow{\beta} b)\}$$

So, The map $r_! h_!(F)(b)_n \to F(b)_n$ is given by $F(\beta\alpha_n...\alpha_1)(x) \in F(b)_n$.

(Step 2) Reduction via colimits: Now assume that F is projectively cofibrant, which means it can be written as a colimit of object of the form $\Delta^n \times Hom(a, -)$. As both are left adjoints and thus commute with colimits, it hence suffices to check the equivalence for the case $\Delta^n \times Hom(a, -)$.

(Step 3) Reduction via initial points: Next notice we have following commutative diagram:

$$\begin{array}{ccc} r_! h_!(\Delta^n \times Hom(a,-)) & \longrightarrow & \Delta^n \times Hom(a,-) \\ {\scriptstyle r_! h_!(\{0\} \times id)} \big\uparrow \simeq & & \big\uparrow {\scriptstyle \{0\} \times id} \\ r_! h_!(\Delta^0 \times Hom(a,-)) & \longrightarrow & \Delta^0 \times Hom(a,-) \end{array}$$

The vertical maps are both projective equivalences as they come from the map $\{0\} : \Delta^0 \to \Delta^n$, which is a covariant trivial cofibration, which is preserved by left Quillen functors. Thus it suffices to prove that the map is an equivalence.

(Step 4) Compute $r_! h_!(Hom(a, -))$: We can finally end the proof by directly computing $r_! h_!(Hom(a, -))$. By the description above we note that

$$r_! h_!(Hom(a, -))(b)_n = \{(\alpha_0 : a \to a_0, a_0 \xrightarrow{\alpha_1} ... \xrightarrow{\alpha_n} a_n \xrightarrow{\beta} b)\} \cong N(A_{a//b})_n$$

The proof the follows from the fact that the projection map:

$$N(A_{a//b}) \to Hom(a, b)$$

is a Kan equivalence, as for any given map $f : a \to b$ the fiber is the full subcategory of $A_{a//b}$ that composes to f and that has an evident final object, namely $a \xrightarrow{f} b \xrightarrow{=} b$. □

Remark 6.18. From the perspective of the proof we can think of $r_! h_!(F)$ as a "fattening up" of F (assuming F is cofibrant) that replaces the various hom sets with the space of all possible compositions.

We now move on to the last proposition.

Proposition 6.19 ([HM15, Proposition 5.2])**.** *For any map of simplicial sets $X \to NA$ there exists a natural zigzag*

$$X \xrightarrow{\iota} X \times_{NA} NA^{\Delta^1} \xleftarrow{\gamma} h_! r_!(X)$$

of covariant weak equivalences over NA.

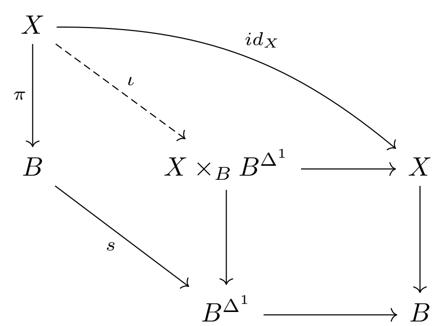
Proof. To simplify notation we denote $B = NA$. As before the proof has many steps:

(Step 1) Alternative characterization of $h_!r_!(X)$: Before we can do anything we need a better understanding of the fibration $\pi : h_!r_!(X) \to B$. An n-simplex in $h_!r_!(X)$ is given by a tuple

$$\{(x, \pi(x_n) \to a_0 \to ... \to a_n) | x \in X_n, x_n \text{ final vertex }\}$$

The projection map π takes such a tuple to the chain $a_0 \to ... \to a_n$.

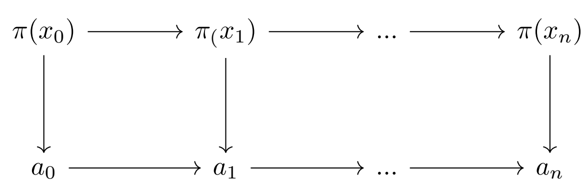
(Step 2) Defining ι and γ: We now want to define ι and γ. Define ι via the diagram:



On the other hand define γ as the map that sends a point

$$\{(x, \pi(x_n) \to a_0 \to ... \to a_n) | x \in X_n, x_n \text{ final vertex }\}$$

to the the pair x and the diagram in A:



where the horizontal maps are given by definition and the vertical maps are suitable compositions taken from the sequence

$$\pi(x_0) \to ... \to \pi(x_n) \to a_0 \to ... \to a_n$$

(Step 3) Proving ι is a covariant equivalence: We are now in a position to prove that ι is a covariant equivalence. We will do so by constructing a deformation retract. Note there is a deformation retract

$$B^{\Delta^1} \times \Delta^1 \to B^{\Delta^1}$$

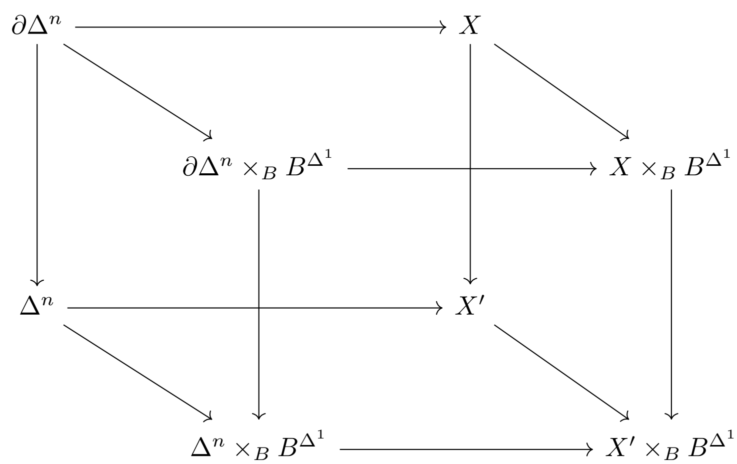
from $con \circ ev_0 \Rightarrow id$ that prove the maps

$$B^{\Delta^1} \xrightarrow{ev_0} B \xrightarrow{con} B^{\Delta^1}$$

are covariant equivalences. Pulling this diagram back along $\pi : X \to B$ gives us the desired deformation retract of the maps

$$B^{\Delta^1} \times_B X \to X \to B^{\Delta^1} \times_B X$$

(Step 4) Reducing the proof for γ to simplices: In order to prove that γ is a covariant equivalence we first reduce it to the case where $X = \Delta^n$. Indeed we have following diagram:



This diagram is a homotopy pushout square as all objects are cofibrant one side is mono and so the map $X' \to X' \times_B B^{\Delta^1}$ is a covariant equivalence if and only if the three other maps are, which justifies our induction approach.

(Step 5) Reducing it to the initial point: We want to further reduce the argument from a simplex to a point. Let $\beta : \Delta^n \to B$ be a simplex and $b_0 : \Delta^0 \to B$ be the initial vertex of β. Recall that the inclusion $b_0 \hookrightarrow \Delta^n \to B$ is a covariant equivalence over B. Then we have following diagram:

$$\begin{array}{ccc} \Delta^0 \times_B B^{\Delta^1} & \longleftarrow & h_! r_! (\Delta^0) \\ \downarrow \simeq & & \downarrow \simeq \\ \Delta^n \times_B B^{\Delta^1} & \longleftarrow & h_! r_! (\Delta^n) \end{array}$$

Both $h_! r_!$ and $(-) \times_B B^{\Delta^1}$ preserve covariant equivalences. The first because $h_! r_!$ is left Quillen and the second because $B^{\Delta^1} \to B$ is a covariant equivalence. Thus the top map is an equivalence if and only if the bottom map is an equivalence.

(Step 6) Use a 2-out-of-3 argument: Now notice we have following diagram:

$$\begin{array}{ccc} & b_0 & \\ \swarrow & & \searrow^{\simeq} \\ h_! r_! (b_0) & \xrightarrow{\gamma} & \Delta^0 \times_B B^{\Delta^1} \end{array}$$

We already know that the right leg is an equivalence, thus in order to prove that γ is an equivalence it suffices to prove the map

$$b_0 \to h_! r_! (b_0)$$

is a covariant equivalence over B.

(Step 7) Apply the Yoneda lemma: In the last step we deviate from the original proof and just compute $h_! r_! (b_0)$. As described above an n-cell is given by tuples

$$\{b_0, b_0 \to a_0 \to ... \to a_n\}$$

Thus $h_! r_! (b_0) = N(B_{b_0/})$ and the map $b_0 \to h_! r_! (b_0)$ picks out the identity map. However, this is in fact a covariant equivalence, by the Yoneda lemma. □

That's it! We have hence proven a Quillen equivalence between left fibrations over NA and functors out of A. In particular, if we take $A = [n]$ then $NA = \Delta^n$ and so we get a classification of left fibrations over Δ^n recovering the examples we discussed before.

One elegant aspect of the proof is that $h_!(F)$ is literally given as a level-wise Grothendieck construction on 1-categories. So, it gives us a very concrete construction of "unstraightening".

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