Elementary Higher Topos and Natural Number Objects

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The goal of the talk is to look at some ongoing work about logical phenomena in the world of spaces. Because I am assuming the room has a topology background I will therefore first say something about the logical background and then move towards topology.

1. Set Theory and Elementary Toposes
2. Natural Number Objects and Induction
3. Elementary Higher Topos
4. Natural Number Objects in an Elementary Higher Topos
5. Where do we go from here?

Set Theory and Elementary Toposes

A lot of mathematics is built on the language of set theory. A common way to define a set theory is via ZFC axiomatization. It is a list of axioms that we commonly associate with sets. Here are two examples:

1. **Axiom of Extensionality**: S = T if $z \in S \Leftrightarrow z \in T$.
2. **Axiom of Union**: If S and T are two sets then there is a set $S \cup T$ which is characterized as having the elements of S and T.

This approach to set theory was developed early $20th$ century and using sets we can then define groups, rings and other mathematical structures.

Later the language of category theory was developed which motivates us to study categories in which the objects behave like sets. Concretely we can translate the set theoretical conditions into the language of category theory. For example we can translate the conditions above as follows:

1. **Axiom of Extensionality**: The category has a final object 1 and it is a generator.
2. **Axiom of Union**: The category is closed under coproducts.

The study of such categories and other important examples let to the study of an elementary topos.

Definition 1.1. An *elementary topos* $\mathcal{E}$ is a locally Cartesian closed category with subobject classifier.

Definition 1.2. $\mathcal{E}$ is locally Cartesian closed if it has finite limits and for any map $f : x \to y$ the pullback map

$$f^* : \mathcal{E}_{/y} \to \mathcal{E}_{/x}$$

has a right adjoint. We can think of that right adjoint as an internal mapping object.

Definition 1.3. Let

$$Sub(-) : \mathcal{E}^{op} \to \mathcal{S}\text{et}$$

be the functor that takes an object c to the set of equivalence classes of subobjects of c, Sub(c). An object Ω is a *subobject classifier* if it represents the functor $Sub(-)$.

In particular, this implies that for any mono map $i : B \to A$ there exists a pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle t} \\ A & \longrightarrow & \Omega \end{array}$$

Thus $t : 1 \to \Omega$ is the universal monomorphism.

As you might have noticed an elementary topos does not satisfy all conditions we expect sets to satisfy (for example the condition that it is generated by the final object). However, it serves as a broad definition which can be restricted based on need.

The key thing about an elementary topos (if we want to use it to define sets) is that we must be able to phrase all conditions in a finite way and cannot use infinite sets. So, for example, the existence of countable colimits is not an elementary condition.

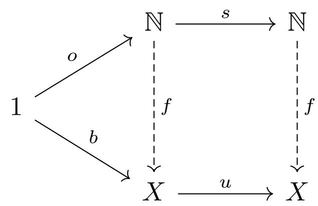
Natural Number Objects and Induction

Here is an example of an elementary topos.

Example 2.1. Let $\mathcal{S}\mathrm{et}_f$ be the category of finite sets. It is a topos as it is locally Cartesian closed and has a subobject classifier, namely the map $\{1\} \to \{0, 1\}$.

Finite sets already have many characteristics we would expect of sets (such as generated by the point, axiom of choice, finite colimits and limits, ...) except for that fact that we do not have infinite cardinals. As we mentioned before we cannot have infinite coproducts so we have to force the existence of an infinite cardinal in a different way. That leads us to the study of natural number objects.

Definition 2.2. Let $\mathcal{E}$ be an elementary topos. A natural number object is the data $1 \to \mathbb{N} \to \mathbb{N}$ such that for each map $1 \to X \to X$ there exists a unique map $f : \mathbb{N} \to X$ such that following diagram commutes:



Notice that the universal condition implies that if a natural number object exists then it is certainly unique.

Example 2.3. In the category of sets the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ along with the map $o : \{0\} \to \mathbb{N}$ and the successor map $s(n) = n+1$ is a natural number object. Concretely every map $f : \mathbb{N} \to X$ is then just a sequence and the unique sequence is defined as $a_0 = b$ and $a_{n+1} = f(a_n)$.

This can be generalized.

Example 2.4. Let $\mathcal{G}$ be a Grothendieck topos, meaning a category of set valued sheaves on a small site. Then $\mathcal{G}$ has a natural number object namely the sheafification of the of the constant presheaf valued in the set $\mathbb{N}$.

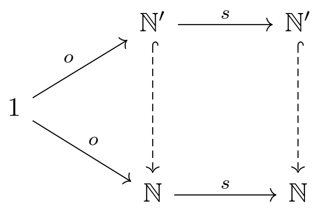
Example 2.5. More generally, let $\mathcal{E}$ be an elementary topos with countable colimits. Then the infinite coproduct $\coprod_{\mathbb{N}} 1$ is a natural number object.

Remark 2.6. Notice the reverse does not hold. We can have an elementary topos $\mathcal{E}$ without countable colimits but that have a natural number object. However, example are always difficult to construct [Jo03, Example D5.1.7].

So, studying natural number objects is only interesting in the case of elementary toposes that are not Grothendieck.

Thus a natural number object allows us to define a notion of infinity in a finite way. It can serve as a substitute to actual infinite structure. In particular natural number objects give us a notion of induction:

Theorem 2.7. *$\mathbb{N}'$ be a subobject of $\mathbb{N}$ such that we have following diagram*



Then $\mathbb{N}' = \mathbb{N}$.

Remark 2.8. This is the categorification of mathematical induction. In fact the Peano axioms (of which induction is one) are equivalent to the original definition. (The other ones are the existence of the maps s and o, the map s is mono and o and s are disjoint).

So, what are some concrete benefits of natural number objects?

1. **Free Algebras**:

   Theorem 2.9. *[Jo03, Subsection D5.3] Let $Mon(\mathcal{E})$ be the category of monoid objects in $\mathcal{E}$. Then $\mathcal{E}$ has a natural number object if and only if the forgetful functor $U : Mon(\mathcal{E}) \to \mathcal{E}$ has a left adjoint. Meaning that we can construct free monoids.*

   Remark 2.10. If we have infinite coproduct and products then for each object X we can define
   $$\coprod_{n \in \mathbb{N}} X^n$$
   which has a monoidal structure taking two lists $(x_0, ..., x_m)$, $(x'_0, ..., x'_{m'})$ to the list $(x_0, ..., x_m, x'_0, ..., x'_{m'})$ and where the identity is the empty list. Then we can show that this monoid is the free monoid on X.

   A natural number object allows us to define the object that behaves similar to the object $\coprod_{n \in \mathbb{N}} X^n$ in an internal way without the need for any infinite coproduts.

   Remark 2.11. If we have free monoids then the free monoid on the final object will be the natural number object.

   We can generalize this argument to all kinds of free algebraic structures, for example groups.

2. **Set Theories**:

   Finally we can use natural number objects to build models of set theory:

   Theorem 2.12. *[MM12, Subsection VI.10] We can build a model for restrictive Zermelo Frankel set theory out of an elementary topos which is generated by the final object and has a natural number object.*

Elementary Higher Toposes

As a homotopy theorist I am interested in foundational questions about spaces. We have different ways to approach spaces and algebraic topology, most notably topological spaces or Kan complexes. The question I have been wondering is how we can axiomatize a theory of spaces the same way we have done for sets, with the hope of separating a notion of space from its set theoretical roots. The plan is to characterize the category of spaces by giving internal conditions on the category.

The first step is to recognize that we should really work with a model of $(\infty, 1)$-category, which I will also call higher category. This because we are concerned with the homotopy thory of spaces which can be best studied in a higher category, rather than a classical category.

As an elementary topos was a very helpful first step towards understanding the category of sets, one analoguous first step is to understand a higher categorical generalization. That is why I have focused on the study of *elementary higher toposes*.

Definition 3.1. An *elementary higher topos* is a higher category $\mathcal{E}$ that satisfies four conditions:

1. $\mathcal{E}$ has finite limits and colimits
2. $\mathcal{E}$ has a subobject classifier
3. $\mathcal{E}$ is locally Cartesian closed.
4. $\mathcal{E}$ has universes.

Example 3.2. Every *higher topos* (meaning sheaves of spaces) is an elementary higher topos.

Proposition 3.3. *The subcategory of* 0*-truncated objects of an elementary higher topos* $\mathcal{E}$ *is an elementary topos.*

Elementary Higher Toposes and Natural Number Objects

Similar to the case of elementary toposes many inductive and infinite constructions are not possible in the current setting, which why we need a natural number object. However, instead of assuming their existence we want to show it exists. How can we do that? The goal is to construct an infinite object based on a finite construction.

Turns out the answer can be found in any algebraic topology textbook: We can always form following finite diagram:

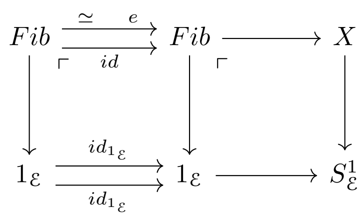
$$\begin{array}{ccccc} \Omega S^1 & \overset{\simeq}{\underset{\simeq}{\rightrightarrows}} & \Omega S^1 & \longrightarrow & 1 \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ 1 & \rightrightarrows & 1 & \longrightarrow & S^1 \end{array}$$

In spaces we know that $\Omega S^1 = \mathbb{Z}$, which is an inherently infinite object. The goal is to use this fact to find a natural number object in a elementary higher topos.

The question is which properties hold in every elementary higher topos and which ones can and cannot be generalized? Let $S^1_{\mathcal{E}}$ be the coequalizer of the final diagram. Clearly we cannot say $\Omega S^1_{\mathcal{E}} = \coprod_{\mathbb{Z}} 1$ as we do not have infinite colimits. However, every elementary higher topos satisfies descent, which in particular implies that we have an equivalence

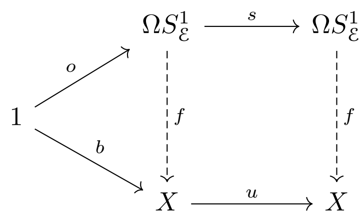
$$\mathcal{E}^{S^1} \simeq \mathcal{E}_{/S^1_{\mathcal{E}}}$$

which relates a map $X \to S^1_{\mathcal{E}}$ to its pullback (Fib, e).



This equivalence has several direct implications:

1. $\Omega S^1_{\mathcal{E}}$ is 0-truncated
2. $\Omega S^1_{\mathcal{E}}$ is a group object (the representable functor is a loop space).
3. $\Omega S^1_{\mathcal{E}}$ is a free group generated by the final object.
4. Even more generally we have following: For every equivalence $u : X \to X$ and point $b : 1 \to X$.



There is unique $f : \Omega S^1_{\mathcal{E}} \to X$ making the digram commute.

Using this structure we can construct the smallest subobject of $\Omega S^1_{\mathcal{E}}$ closed under the successor and 0 map, which we call $\mathbb{N}$.

Immediately we get that:

Proposition 4.1. $\mathbb{N}$ *satisfies induction.*

Theorem 4.2. *(Quasi-Theorem) The object* $\mathbb{N}$ *is a natural number object.*

Remark 4.3. One reason for caution is that it is not immediate that various conditions on natural number objects that we know are equivalent in a classical category are still equivalent in the higher categorical setting.

Where do we go from here?

Here are three things that come to mind:

1. First of all it is interesting to notice the philosophical implications. In the world of elementary toposes natural number objects don't have to exist. On the other side in the world of elementary higher toposes infinite structures exist by default. The homotopical data necessitates the existence. That raises the question what other structures we might be able to construct, that we would not expect to find in an elementary topos.

2. On a more concrete level, given that in the classical setting we were able to use natural number objects to construct free algebras, the next step would be to show that we can construct free algebras in every elementary higher topos.

3. Finally, one property that makes spaces very interesting is the existence of truncations. It allows us to decompose spaces into smaller pieces that we can then try to patch together. Another aim is to show that we can use a natural number object to construct truncations of objects.

References

[Jo03] P. Johnstone, *Sketches of an elephant: A topos theory compendium-2 volume set.* pp. 1288. Foreword by Peter T Johnstone. Oxford University Press, Jul 2003. ISBN-10:. ISBN-13: 9780198524960 (2003): 1288.

[MM12] S. MacLane,I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory.* Springer Science & Business Media, 2012.