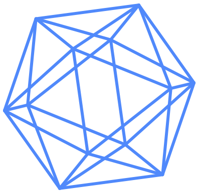
An Axiomatic Approach to Algebraic Topology

Nima Rasekh

Max-Planck-Institut für Mathematik



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Overview

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Question

What does axiomatic mean? Why should we do anything axiomatic? How can we do something axiomatic?

What is a Set - Appearance

What is a set?

Definition

A set is a left curly bracket, an element, a comma symbol, another element, another comma symbol, ... and finally a right curly bracket. For example:

$$\{1, a, \sim, \&, \text{topology}\}$$

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This doesn't sound right!
When using sets, we don't primarily care about how they **look like** but rather how they **behave**.

Axioms of Set Theory

We want our sets to satisfy some rules:

1. **Axiom of Union:** If A and B are two sets then there exists a set $A \cup B$ whose elements consists of elements in A and B.
2. **Axiom of Function Extensionality:** If $f, g : A \to B$ are two functions then
$$f = g \Leftrightarrow \forall x \in A(f(x) = g(x))$$
3. ...

But where do these axioms hold?

Logic vs. Category Theory

There are various ways to make sense of symbols such as $\forall$ or $x \in A$.

1. **First-Order Logic**
2. **Category Theory**

ZFC vs. Elementary Topos

Each method gives us one way to axiomatize sets:

1. **Zermelo-Frankel Set Theory**: A set is an element in a model of the ZF axioms.
2. **Elementary Topos Theory**: A set is an object in an elementary topos.

Thus we have described a set by what we want it to do and not how it looks like!

What is a space?

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You might say ...

1. *Kan complex (CW-complex)*
2. *Topological space*

Again, we are just saying how a space looks like rather than how it behaves.
How do we fix this?

Two Paths Towards Axiomatization of Spaces

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We want to exclusively focus on the categorical approach: **higher category theory**

Higher Category Theory

A *higher category* $\mathcal{C}$ has following properties:

1. It has objects $x, y, z, \ldots$.
2. For any two objects x, y there is a mapping space (Kan complex) $Map_{\mathcal{C}}(x, y)$ with a notion of composition that holds only "up to homotopy".
3. This is a direct generalization of classical categories and all categorical notions (limits, adjunction, ...) generalize to this setting.

Examples

1. Small Kan complexes form a higher category which we denote by $\mathcal{K}an$. Similarly, $\widehat{\mathcal{K}an}$ denotes the higher category of large Kan complexes.
2. Classical categories are all higher categories, in particular $\mathcal{S}et$ is a higher category.

Axiomatizing Spaces via Higher Categories

Henceforth the framework is higher categories. Thus our axiomatization should be a higher category which satisfies certain conditions. In analogy with the classical setting we call it an *elementary higher topos*.

Elementary Higher Topos

Definition

We call a higher category $\mathcal{E}$ an *elementary higher topos* if it satisfies following conditions:

1. It has finite limits and colimits.
2. It has a subobject classifier.
3. It is locally Cartesian closed.
4. There exists a chain of *universes* $\{\mathfrak{U}^S\}$ such that each map is classified by a universe.

What is a Universe?

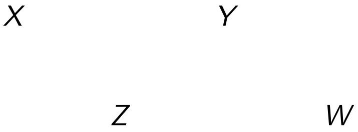
A *universe* (can also be called *object classifier* or *moduli object*) can be thought of as an object in which each point classifies some other object and it always comes with a *universal fibration* $p : \mathcal{U}_* \to \mathcal{U}$ such that for any $f : X \to Y$ we have a pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{U}_* \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\ X & \longrightarrow & \mathcal{U} \end{array}$$

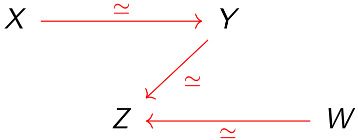
A Universe of Kan Complexes

Let $\mathcal{K}an^{core}$ be following **large** Kan complex:

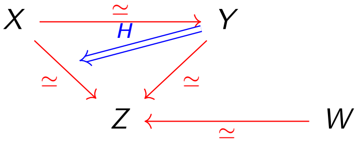
1. **Points (0-Simplices):**



2. **Paths (1-Simplices):**



3. **Two-Simplices:**



A Univeral Fibration of Kan Complexes

We can build a similar space using pointed Kan complexes and pointed equivalences that we call $(\mathcal{K}an_*)^{core}$ which comes with a forgetful map $U : (\mathcal{K}an_*)^{core} \to \mathcal{K}an^{core}$. This map is a universal fibration for small spaces.

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$$\begin{array}{ccc} X \simeq Map(*, X) & \longrightarrow & (\mathcal{K}\mathrm{an}_*)^{core} \\ \big\downarrow & \ulcorner & \big\downarrow U \\ * & \xrightarrow[\{X\}]{} & \mathcal{K}\mathrm{an}^{core} \end{array}$$

Examples

Example

The higher category of large Kan complexes $\widehat{\mathcal{K}an}$ is an elementary higher topos.

Example

Every *Grothendieck higher topos* (in the sense of Lurie) is also an elementary higher topos.

Classical Result in an Elementary Higher Topos

In analogy with set theory a homotopy theorist shouldn't primarily care about the definition but rather how they can use it.
So, let's see some examples ...

Truncated Objects

Definition

A space X is n-truncated if $\pi_k(X) = *$ for all $k > n$.

Definition

An object X in a higher category is n-truncated if $Map(Y, X)$ is n-truncated for all Y.

Truncations

One amazing feature of spaces is the existence of truncations. We can take any space X and universally construct a truncated object $\tau_n X$.

Theorem

There exists an adjunction

$$\mathcal{K}an \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n \mathcal{K}an$$

where i is the inclusion.

Truncations in an Elementary Higher Topos

Theorem (R.)

Let $\mathcal{E}$ be an EHT. Then there exists an adjunction

$$\mathcal{E} \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n\mathcal{E}$$

where i is the inclusion.

The idea for the proof comes from work of Egbert Rijke in the context of homotopy type theory.

Connected Maps

Definition

A map of objects $f : X \to Y$ is *n*-connected if for any *n*-truncated object Z the map

$$f^* : Map(Y, Z) \to Map(X, Z)$$

is an equivalence of spaces.

Blakers-Massey Theorem

Another fundamental result in algebraic topology is the Blakers-Massey theorem, which holds analogously for EHT.

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Theorem (R.)

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & W \end{array}$$

Let the above be a pushout diagram in an EHT $\mathcal{E}$, such that f is m-connected and g is n-connected. Then the map $(f, g) : Z \to X \times_W Y$ is $(m + n)$-connected.

This proof is based on work by Anel, Biedermann, Finster and Joyal on higher toposes.

Further Algebraic Topology in an Elementary Higher Topos

What else can we do? Here are some further topics related to algebraic topology that can be studied in an EHT:

1. We have truncations and spheres, which means we can define homotopy groups. How does the homotopy groups of spheres compare with the classical homotopy groups?
2. Blakers-Massey gives us Freudenthal suspension theorem, which means we have stabilizations. How does the stabilization compare to spectra?
3. Can we construct Eilenberg-MacLane objects in an elementary higher topos?