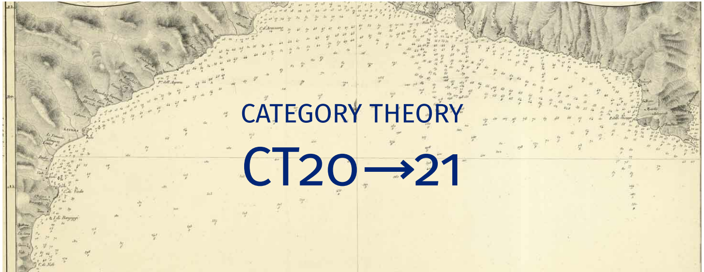
Fibrations of (∞, n)-Categories

Nima Rasekh

Ecole Polytechnique Fédérale de Lausanne



More Information:

- **ArXiv:**
  - *Yoneda lemma for $\mathcal{D}$-simplicial spaces*
  - arXiv:2108.06168.
- **Academic Website:**

  https : //www.epfl.ch/labs/hessbellwald − lab/members/nimarasekh/

- **Email:** nima.rasekh@epfl.ch

Functors vs. Fibrations

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}) \underset{\int_{\mathcal{C}}}{\overset{\simeq}{\rightleftarrows}} \mathrm{op}\mathcal{G}\mathrm{roth}_{/\mathcal{C}}$$

$$F \longmapsto \coprod_{c \in \mathrm{Obj}_{\mathcal{C}}} F(c)$$

$$p^{-1}(c) \longleftarrow\!\!\mid\; p$$

Examples and Yoneda

- Constant functors $\{A\} : \mathcal{C} \to \mathcal{S}et$ correspond to projections $A \times \mathcal{C} \to \mathcal{C}$.
- Representables $\mathrm{Hom}(c, -) : \mathcal{C} \to \mathcal{S}et$ correspond to under-categories $\mathcal{C}_{c/} \to \mathcal{C}$.

Examples and Yoneda

- Constant functors $\{A\} : \mathcal{C} \to \mathcal{S}\mathrm{et}$ correspond to projections $A \times \mathcal{C} \to \mathcal{C}$.
- Representables $\mathrm{Hom}(c, -) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ correspond to under-categories $\mathcal{C}_{c/} \to \mathcal{C}$.

We also have a Yoneda lemma.

Lemma

For a Grothendieck opfibration $\mathcal{G} \to \mathcal{C}$ and object c there is an isomorphism

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{G}) \cong \mathcal{F}\mathrm{ib}_c \mathcal{G}$$

On to $(\infty, 1)$-Categories

- Categories generalize to $(\infty, 1)$-categories!
- For an $(\infty, 1)$-category $\mathcal{C}$, we often cannot define functors, such as

  $$\mathrm{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathcal{S}$$

  as composition is defined *weakly*.
- We *need* the fibrational approach to study functors.

coCartesian Fibrations of $(\infty, 1)$-Categories

The fibrational approach is obtained via *coCartesian (left) fibrations*. For a given $(\infty, 1)$-category $\mathcal{C}$ we have equivalences

$$\mathrm{co}\mathcal{C}\mathrm{art}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$$

$$\mathcal{LF}\mathrm{ib}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$$

coCartesian Fibrations of $(\infty, 1)$-Categories

The fibrational approach is obtained via *coCartesian (left) fibrations*. For a given $(\infty, 1)$-category $\mathcal{C}$ we have equivalences

$$\text{co}\mathcal{C}\text{art}_{/\mathcal{C}} \simeq \text{Fun}(\mathcal{C}, \mathcal{C}\text{at}_\infty)$$

$$\mathcal{L}\mathcal{F}\text{ib}_{/\mathcal{C}} \simeq \text{Fun}(\mathcal{C}, \mathcal{S})$$

Studied in different ways by *Lurie*, *Heuts-Moerdjik*, *Stevenson*, *Riehl-Verity*, *deBrito*, *Ayala-Francis*, ...

coCartesian Fibrations of $(\infty, 1)$-Categories

The fibrational approach is obtained via *coCartesian (left) fibrations*. For a given $(\infty, 1)$-category $\mathcal{C}$ we have equivalences

$$\mathrm{co}\mathcal{C}\mathrm{art}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_{\infty})$$

$$\mathcal{L}\mathcal{F}\mathrm{ib}_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$$

Studied in different ways by *Lurie*, *Heuts-Moerdjik*, *Stevenson*, *Riehl-Verity*, *deBrito*, *Ayala-Francis*, ...

Similar to 1-categories, representable functors $\mathrm{Map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathcal{S}$ are obtained via $\mathcal{C}_{c/} \to \mathcal{C}$.

Fibrations are Useful!

We use *coCartesian fibrations* to study many ∞-categorical concepts:

- Limits, colimits, adjunctions
- Symmetric monoidal ∞-categories
- ∞-operads
- ...

The (∞, n)-Categorical World

$(\infty, 1)$-categories have been generalized to several models of (∞, n)-categories

- n-fold complete Segal spaces
- Θ_n-spaces
- n-complicial sets
- n-comical sets
- ...

n-fold Complete Segal Spaces

We introduce *Cartesian fibrations* of n-fold complete Segal spaces.

n-fold Complete Segal Spaces

We introduce *Cartesian fibrations* of n-fold complete Segal spaces.

An n-fold complete Segal space $\mathcal{C} : (\Delta^{op})^n \to \mathcal{S}$ satisfying

- **Segal Condition:** for composition
- **Completeness Condition:** for equivalences
- **Discreteness Condition:** to avoid additional morphisms

It comes with a model structure that is known to be equivalent to Θ_n-spaces (Bergner-Rezk) and for $n = 2$ to all other models of $(\infty, 2)$-categories.

Cartesian Fibrations of n-fold Complete Segal Spaces I

Let $\mathcal{C}$ be an $n+1$-fold complete Segal space. There is a notion of (∞, n)*-coCartesian fibration* with the following properties:

Cartesian Fibrations of n-fold Complete Segal Spaces I

Let $\mathcal{C}$ be an $n+1$-fold complete Segal space. There is a notion of (∞, n)-*coCartesian fibration* with the following properties:

1. **Model Structure:** It is a fibrant object in a simplicial left proper combinatorial model structure over $\mathcal{C}$.

Cartesian Fibrations of n-fold Complete Segal Spaces I

Let $\mathcal{C}$ be an $n+1$-fold complete Segal space. There is a notion of (∞, n)-*coCartesian fibration* with the following properties:

1. **Model Structure:** It is a fibrant object in a simplicial left proper combinatorial model structure over $\mathcal{C}$.
2. **Yoneda Lemma:** For an object c in $\mathcal{C}$, there is a *representable* (∞, n)-*coCartesian fibration* $\mathcal{C}_{c/} \to \mathcal{C}$ and for every (∞, n)-coCartesian fibration $\mathcal{L} \to \mathcal{C}$ and object c in $\mathcal{C}$ we have an equivalence of (∞, n)-categories
$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{L}) \simeq \mathcal{F}\mathrm{ib}_c\mathcal{L}.$$

Cartesian Fibrations of n-fold Complete Segal Spaces I

Let $\mathcal{C}$ be an $n+1$-fold complete Segal space. There is a notion of (∞, n)-*coCartesian fibration* with the following properties:

1. **Model Structure:** It is a fibrant object in a simplicial left proper combinatorial model structure over $\mathcal{C}$.
2. **Yoneda Lemma:** For an object c in $\mathcal{C}$, there is a *representable* (∞, n)-*coCartesian fibration* $\mathcal{C}_{c/} \to \mathcal{C}$ and for every (∞, n)-coCartesian fibration $\mathcal{L} \to \mathcal{C}$ and object c in $\mathcal{C}$ we have an equivalence of (∞, n)-categories
$$\text{Fun}_{/\mathcal{C}}(\mathcal{C}_{c/}, \mathcal{L}) \simeq \mathcal{F}\text{ib}_c\mathcal{L}.$$
3. **Dwyer-Kan Equivalence:** A map of (∞, n)-coCartesian fibrations $\mathcal{L} \to \mathcal{L}'$ is an equivalence if and only if it is a Dwyer-Kan equivalence of $(\infty, n+1)$-categories.

Cartesian Fibrations of n-fold Complete Segal Spaces II

4. **Invariance**: The model structure, and hence (∞, n)-coCartesian fibrations are invariant under equivalences of $n+1$-fold complete Segal spaces $\mathcal{C} \to \mathcal{D}$.

Cartesian Fibrations of n-fold Complete Segal Spaces II

4. **Invariance**: The model structure, and hence (∞, n)-coCartesian fibrations are invariant under equivalences of $n+1$-fold complete Segal spaces $\mathcal{C} \to \mathcal{D}$.
5. **Grothendieck Construction:** There is an equivalence

$$(\mathrm{co}\mathcal{C}\mathrm{art}_{(\infty,n)})_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_{(\infty,n)})$$

Cartesian Fibrations of n-fold Complete Segal Spaces II

4. **Invariance**: The model structure, and hence (∞, n)-coCartesian fibrations are invariant under equivalences of $n+1$-fold complete Segal spaces $\mathcal{C} \to \mathcal{D}$.
5. **Grothendieck Construction:** There is an equivalence

$$(\mathrm{co}\mathcal{C}\mathrm{art}_{(\infty,n)})_{/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_{(\infty,n)})$$

6. **Universality and Univalence:** There is a universal (∞, n)-coCartesian fibration that is univalent.

Generalities and the Future

Generalities:

- All these result hold for many other models of (∞, n)-Categories, such as **complete Segal spaces enriched over Θ_n-spaces**.
- They also hold for more general fibrations, such as fibrations **fibered in n-fold Segal spaces**.

Generalities and the Future

Generalities:

- All these result hold for many other models of (∞, n)-Categories, such as **complete Segal spaces enriched over Θ_n-spaces**.
- They also hold for more general fibrations, such as fibrations **fibered in n-fold Segal spaces**.

Future:

- Study **limits** of $(\infty, 2)$-categories via fibrations of 2-fold complete Segal spaces.
- Study $(\infty, 2)$**-topos theory** via representable $(\infty, 2)$-Cartesian fibrations.

Università
di Genova